ANNIHILATOR IDEALS IN 0-DISTRIBUTIVE LATTICES

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Abstract. In this paper we generalized some results of annihilator ideals for 0-distributive lattices. We prove that the set of all annihilator ideals of a 0-distributive lattice form a Boolean algebra. We also prove Stone type separation theorem for annihilator ideals of 0-distributive lattices.

Key words and phrases. 0-distributive lattice, annihilator, ideal, Boolean algebra.

1. Introduction

A lattice $L$ with 0 is called 0-distributive if for any $a, b, c \in L$ such that $a \land b = 0 = a \land c$ implies $a \land (b \lor c) = 0$. Varlet [7] introduced the notion of 0-distributive lattice for generalization of distributive lattices. For the background of lattice theory we refer the reader to the foundation monograph [3].

Annihilator ideals play an important role to study lattice theory. Cornish [2] and Davey [1] studied annihilator ideals for distributive lattices. In this paper we study annihilator ideals of a 0-distributive lattice. Here we generalize some results for 0-distributive lattices.

In Section 2, we prove some identities for annihilators which we need in this paper.

Pawar and Khopade [5] have studied $\alpha$-ideals and annihilator ideals of a 0-distributive lattice as a consequence study of [6, 4]. They have mentioned that the set of all annihilator ideals of a 0-distributive lattice form a Boolean algebra. The lattice of all annihilator ideals is a sublattice of the lattice of all $\alpha$-ideals of a 0-distributive lattice. In Section 3, we discuss annihilator ideals of a 0-distributive lattice. We prove that the set of all annihilator ideals of a 0-distributive lattice...
form a Boolean algebra. Our definition of supremum of two annihilator ideals is different from the definition of supremum of two \( \alpha \)-ideals given in [5].

In Section 4, we found that prime ideals and annihilator ideals of a 0-distributive lattice are independent. In this section we show that a non-dense prime ideal of a 0-distributive semilattice is an annihilator ideal. We also give a characterization of prime ideal in 0-distributive lattices to be an annihilator ideals. We establish a sufficient condition for an annihilator ideal to be a prime ideal. We prove Stone type separation theorem for annihilator ideals of 0-distributive lattices.

### 2. Annihilators

Let \( L \) be a 0-distributive lattice and \( A \) be a non-empty subset of \( L \). Define

\[
A^\perp = \{ x \in S \mid x \land a = 0 \text{ for all } a \in A \}.
\]

Then \( A^\perp \) is called the annihilator of \( A \). If \( a \in A \), then the annihilator of \( \{a\} \) is denoted by \( a^\perp \) and defined as

\[
a^\perp = \{ x \in L \mid x \land a = 0 \}.
\]

It is called the annulet generated by \( a \).

The following identities are easy to prove and will be used throughout this paper.

**Lemma 2.1.** Let \( L \) be a lattice with 0 and \( a, b \in L \). Then

(i) \( a \in a^{\perp \perp} \).

(ii) If \( a \nleq b \), then \( b^\perp \subseteq a^\perp \) and \( a^{\perp \perp} \subseteq b^{\perp \perp} \).

(iii) \( a^{\perp \perp} = a^\perp \).

(iv) \( a^\perp \cap a^{\perp \perp} = \{0\} \).

Now we have the following useful result.

**Lemma 2.2.** Let \( L \) be a 0-distributive lattice and \( a, b \in L \). Then

\[
(a \land b)^{\perp \perp} = a^{\perp \perp} \cap b^{\perp \perp}.
\]

**Proof.** Let \( x \in a^{\perp \perp} \cap b^{\perp \perp} \) and \( y \in (a \land b)^{\perp} \). Then \( y \land a \land b = 0 \). This implies \( y \land a \in b^{\perp} \). Since \( x \in b^{\perp \perp} \), therefore \( x \land y \land a = 0 \). This implies \( x \land y \in a^{\perp} \). Since \( x \in a^{\perp \perp} \), therefore \( x \land y \in a^{\perp \perp} \). Thus \( x \land y \in a^{\perp} \cap a^{\perp \perp} = 0 \). This implies \( x \land z = 0 \) for all \( z \in (a \land b)^{\perp} \). Then \( x \in (a \land b)^{\perp \perp} \) and hence \( a^{\perp \perp} \cap b^{\perp \perp} \subseteq (a \land b)^{\perp \perp} \). Converse is due to Lemma 2.1 (ii). Hence \( (a \land b)^{\perp \perp} = a^{\perp \perp} \cap b^{\perp \perp} \).
3. Annihilator Ideals

Let $L$ be a 0-distributive lattice. A non-empty subset $F$ of $L$ is called a filter if (i) $a \in F$ and $b \in L$ with $b > a$ implies $b \in F$, and (ii) $a, b \in F$ implies $a \wedge b \in F$. For any $a \in L$, the set

$$[a] = \{x \in S \mid a \not\leq x\}$$

is a filter called the principal filter generated by $a$.

A non-empty subset $I$ of $L$ is called an ideal if (i) $a \in I$ and $b \in S$ with $b \not\leq a$ implies $b \not\in I$, and (ii) $a, b \in I$ implies $a \vee b \in I$. For any $a \in L$ the set

$$(a) = \{x \in S \mid x \not\in a\}$$

is an ideal called the principal ideal generated by $a$. The set of all ideals of $L$ is denoted by $I(L)$. It is well known that $I(L)$ form a lattice as an ordered set. For $I, J \in I(L)$, we denote $I \wedge J = \inf_{I(L)}\{I, J\}$ and $I \vee J = \sup_{I(L)}\{I, J\}$

Now we have the following useful properties for $I(L)$.

Lemma 3.1. Let $L$ be a lattice with 0 and $I, J \in I(L)$. Then

(i) $I \wedge J = (0)$ if and only if $J \subseteq I^\perp$.

(ii) $I \wedge I^{\perp} = (0)$.

(iii) $I \subseteq I^{\perp\perp}$.

(iv) If $I \subseteq J$, then $J^{\perp} \subseteq I^{\perp}$.

(v) $I^{\perp\perp\perp} = I^{\perp}$.

Proof. (i) Let $b \in J$. Then $a \wedge b = 0$ for all $a \in I$, as $I \wedge J = (0)$. Thus $b \in I^{\perp}$. Hence $J \subseteq I^{\perp}$. Converse is obvious.

(ii) Putting $J = I^{\perp}$ in (i).

(iii) Using (i) and (ii).

(iv) Let $x \in J^{\perp}$. Then $x \wedge j = 0$ for all $j \in J$. Since $I \subseteq J$, therefore $x \wedge i = 0$ for all $i \in I$. Hence $J^{\perp} \subseteq I^{\perp}$.

(v) By (iii), $I^{\perp} \subseteq I^{\perp\perp\perp}$. Converse is true by (iii) and (iv). Hence $I^{\perp\perp\perp} = I^{\perp}$.

Now we have the following useful result for ideals.

Lemma 3.2. Let $L$ be a 0-distributive lattice and $I, J \in I(L)$. Then $(I \wedge J)^{\perp\perp} = I^{\perp\perp} \wedge J^{\perp\perp}$.

Proof. Let $x \in I^{\perp\perp} \wedge J^{\perp\perp}$ and $y \in (I \wedge J)^{\perp}$. Again let $i \in I$ and $j \in J$. Since $i \wedge j \in I \wedge J$ and $y \in (I \wedge J)^{\perp}$, therefore $(y \wedge i) \wedge j = 0$. This
implies $y \land i \in J^\perp$ for all $j \in J$. Hence $y \land i \in J^\perp$. Since $x \in J^{\perp\perp}$, we get $(x \land y) \land i = 0$ for all $i \in I$. Hence $x \land y \in I^\perp$. Since $S$ is 0-distributive, therefore $I^{\perp\perp} \in I(S)$ and hence $x \in I^{\perp\perp}$ implies $x \land y \in I^{\perp\perp}$. Thus $x \land y \in I^\perp \land I^{\perp\perp} = \{0\}$. Hence $x \land z = 0$ for all $z \in (I \land J)^\perp$. Therefore $x \in (I \land J)^{\perp\perp}$. Thus $I^{\perp\perp} \land J^{\perp\perp} \subseteq (I \land J)^{\perp\perp}$. Converse follows from Lemma 3.1(iv). Hence $(I \land J)^{\perp\perp} = I^{\perp\perp} \land J^{\perp\perp}$.

**Lemma 3.3.** Let $L$ be a 0-distributive lattice and $I, J \in I(L)$. Then

$$(I \lor J)^\perp = I^\perp \land J^\perp$$

**Proof.** Since $I, J \subseteq I \lor J$, so we have $(I \lor J)^\perp \subseteq I^\perp \land J^\perp$. On the other hand, let $x \in I^\perp \land J^\perp$. Then $x \land s = 0$ for all $s \in J$. By the property of 0-distributivity of $L$, we have $x \land (s \lor t) = 0$. This implies $x \land i = 0$ for all $i \in I \lor J$. Hence $x \in (I \lor J)^\perp$. Thus $(I \lor J)^\perp = I^\perp \land J^\perp$.

Let $L$ be a 0-distributive lattice. Then $A^\perp$ is an ideal of $L$ for any non-empty subset $A$ of $L$. An ideal $I$ of $L$ is called an annihilator ideal if $I = A^\perp$ for some non-empty subset $A$ of $L$ or equivalently if $I = I^{\perp\perp}$. The set of all annihilator ideals of $L$ is denoted by $A(L)$. If $I, J \in A(L)$, then By Lemma 3.2, $I \land J \in A(L)$. Observe that $I \lor J$ may not be an annihilator ideal. For counterexample, consider the lattice $M$ in Figure 1. Then $I = (a)$ and $J = (b)$ are annihilator ideals but $I \lor J = (c)$ is not an annihilator ideal. Thus $hA(L), \subseteq i$ is not a sublattice of $hl(L), \subseteq i$. Now we show that $hA(L), \subseteq i$ is itself a lattice as an ordered set. For any $I, J \in A(L)$ we denote $I \cup J = \inf_{A(L)}\{I, J\}$ and $I \vee J = \sup_{A(L)}\{I, J\}$.

Indeed, we have the following result.

**Theorem 3.4.** Let $L$ be a 0-distributive lattice. For any $I, J \in A(L)$ we have

(a) $I \cup J = I \land J$;

(b) $I \vee J = (I^\perp \land J^\perp)^\perp$.

**Proof.** (a) Obvious.

(b) Clearly $I^\perp \land J^\perp \subseteq I^\perp, J^\perp$. This implies $I^{\perp\perp} = I \subseteq (I^\perp \land J^\perp)^\perp$ and $J^{\perp\perp} = J \subseteq (I^\perp \land J^\perp)^\perp$. Hence $I \lor J \subseteq (I^\perp \land J^\perp)^\perp$. Now let $K$ be an annihilator ideal such that $I, J \subseteq K$. Then $K^\perp \subseteq I^\perp \land J^\perp$. Since $K$ is an annihilator ideal, this implies $(I^\perp \land J^\perp)^\perp \subseteq K^{\perp\perp} = K$. Hence $I \lor J = (I^\perp \land J^\perp)^\perp$. 
Thus $hA(L)$, $\mathbf{t}$, $\mathbf{u}$ is a lattice.

**Lemma 3.5.** Let $L$ be a 0-distributive lattice and $I, J, K \in A(L)$. Then $(I \mathbf{t} J) \cup K \subseteq I \mathbf{t} (J \cup K)$.

**Proof.** We have

$$I \cup K \cup [I^\perp \cup (J \cup K)] \cup = (0] \text{ and } J \cup K \cup [I^\perp \cup (J \cup K)] \cup = (0]$$

$\implies K \cup I^\perp \cup (J \cup K)^\perp \subseteq I^\perp \text{ and } K \cup I^\perp \cup (J \cup K)^\perp \subseteq J^\perp$

$\implies K \cup I^\perp \cup (J \cup K)^\perp \subseteq I^\perp \cup J^\perp$

$\implies [K \cup I^\perp \cup (J \cup K)^\perp] \cup (I^\perp \cup J^\perp)^\perp = (0] \text{ by Lemma 3.1(i)}$

$\implies I^\perp \cup (J \cup K)^\perp \cup [K \cup (I^\perp \cup J^\perp)^\perp] \cup = (0]$

$\implies K \cup (I^\perp \cup J^\perp)^\perp \subseteq [I^\perp \cup (J \cup K)^\perp]^\perp \text{ by Lemma 3.1(i)}$

Hence by Theorem 3.4, we have $(I \mathbf{t} J) \cup K \subseteq I \mathbf{t} (J \cup K)$.

The following theorem shows that $hA(L), \mathbf{t}, \mathbf{u}$ is a distributive lattice.

**Theorem 3.6.** Let $L$ be a 0-distributive lattice. Then $hA(L), \mathbf{t}, \mathbf{u}$ is a distributive lattice.

**Proof.** For any ideals $I, J, K \in A(L)$,

$$(I \mathbf{t} J) \cup (I \mathbf{t} K) \subseteq I \mathbf{t} [J \cup (I \mathbf{t} K)] \text{ by Lemma 3.5}$$

$$\subseteq I \mathbf{t} I \mathbf{t} (J \cup K) \text{ by Lemma 3.5}$$

$$= I \mathbf{t} (J \cup K)$$

The reverse inclusion is trivial and hence $hA(L), \mathbf{t}, \mathbf{u}$ is a distributive lattice.

Clearly, $0^\perp = L$ and $L^\perp = (0]$. Now we have the following result.

**Theorem 3.7.** Let $L$ be a 0-distributive lattice. Then $hA(L); \mathbf{t}, u^-, (0], L$ is a Boolean Algebra.

**Proof.** We have $hA(L); \mathbf{t}, u, (0], L$ is a bounded distributive lattice. Now for any $I \in A(L)$, we have

$$I \cup I^\perp = (0] \text{ and } I \mathbf{t} I^\perp = (I^\perp \cup I^\perp)^\perp = (0]^\perp = L$$

Hence $hA(L); \mathbf{t}, u^-, (0], L$ is a Boolean Algebra.
4. Separation Theorem for Annihilator Ideals

A prime ideal \( P \) is a proper ideal of \( L \) such that \( a \wedge b \in P \) implies either \( a \in P \) or \( b \in P \). We denote the set of all prime ideals of \( L \) by \( \text{P}_1(L) \). A prime ideal \( P \) is called minimal, if for any prime ideal \( Q \subseteq P \) implies \( P = Q \). A filter \( F \) of \( L \) is called a prime filter if \( F = L \) and \( L \setminus F \) is a prime ideal. It is well known that a filter \( F \) is a maximal filter if and only if \( L \setminus F \) is a minimal prime ideal.

We observed in \( M \) (see Figure 1) that \( I = (0] \) is an annihilator ideal but not prime. On the other hand \( I = (c] \) is a prime ideal but not annihilator ideal. Thus the prime ideals and annihilator ideals of a 0-distributive lattice are independent.

![Figure 1. Two important 0-distributive semilattices](image_url)

Let \( L \) be a 0-distributive lattice. An element \( d \in L \) is called dense if \( d^{\perp} = (0] \). The set of all dense element of \( L \) is denoted by \( D(L) \). Thus

\[
D(L) = \{x \in S \mid x^{\perp} = (0]\}.
\]

It is well known that \( D(L) \) is a filter. An ideal \( I \) of \( L \) is called a dense ideal if \( I^{\perp} = (0] \).

**Lemma 4.1.** Let \( L \) be a 0-distributive lattice. Then a proper annihilator ideal contains no dense element.

**Proof.** Let \( I \) be a proper annihilator ideal of \( L \) and \( x \in I \cap D(S) \). Then \( x^{\perp} = (0] \). This implies \( I^{\perp} = (0] \). Therefore \( I \) is a dense ideal and \( I = I^{\perp\perp} = L \), which contradicts the fact that \( I \) is proper. Therefore \( I \cap D(S) = \emptyset \).

**Theorem 4.2.** If a prime ideal \( P \) of a 0-distributive lattice \( L \) is non-dense, then \( P \) is an annihilator ideal.
Proof. Let $P$ be a prime ideal of $L$. If $P$ is non-dense, then $P^\perp = (0)$. This implies there exists $0 = x \in S$ such that $x \in P^\perp$. Hence $P^\perp \subseteq x^\perp$ and so $P \subseteq x^\perp$. We show that $P = x^\perp$. If not, then let $a \in x^\perp \setminus P$. This implies $a \wedge x = 0 \in P$. Thus $x \in P$ as $P$ is a prime ideal. This shows that $x \in P \cap P^\perp = (0)$, which is a contradiction. Hence $x^\perp = P$. Therefore $P$ is an annihilator ideal.

The above results give us the following corollary.

**Corollary 4.3.** A prime ideal $P$ of a 0-distributive lattice $L$ is an annihilator ideal if and only if $P \cap D(L) = \emptyset$.

Now we have the following theorem.

**Theorem 4.4 (Separation Theorem for Annihilator Ideals).** Let $L$ be 0-distributive lattice, $I$ be an annihilator ideal and $F$ be a filter of $L$ such that $I \cap F = \emptyset$. Then there is a prime annihilator ideal containing $I$ and disjoint from $F$.

**Proof.** Let $I$ be an annihilator ideal and $F$ be a filter of a lattice $L$ such that $I \cap F = \emptyset$. Define

$$X = \{G \mid G \text{ is a filter of } L \text{ such that } F \subseteq G \text{ and } I \cap G = \emptyset\}.$$

Clearly, $F \in X$ and $X$ satisfies all the conditions of Zorn’s Lemma. Thus $X$ has a maximal element, say $M$. We show that $M$ is a prime filter.

Suppose $M$ is not prime. Then there are $a, b \in L \setminus M = P$ such that $a \vee b \in M$. By the maximality of $M$ in $X$, we have $(M \vee \{a\}) \cap I = \emptyset$ and $(M \vee \{b\}) \cap I = \emptyset$. Let $x \in (M \vee \{a\}) \cap I = \emptyset$ and $y \in (M \vee \{b\}) \cap I = \emptyset$. Then $x, y \in I$ such that $x > m \wedge a$ and $y > n \wedge b$ for some $m, n \in M$. Since $I$ is an annihilator ideal, $m \wedge a \in I = I^\perp$ and $n \wedge b \in I = I^\perp$. This implies $m \wedge a \wedge i = 0$ and $n \wedge b \wedge i = 0$ for all $i \in I^\perp$. Hence $(m \wedge n \wedge i) \wedge a = 0$ and $(m \wedge n \wedge i) \wedge b = 0$. Now since $L$ is 0-distributive, we have $(m \wedge n \wedge i) \wedge (a \vee b) = 0$ for all $i \in I^\perp$. Thus $(m \wedge n) \wedge (a \vee b) \in I^\perp = I$. Which is a contradiction to the fact that $I \cap M = \emptyset$. Therefore, $M$ is a prime filter and hence $P = L \setminus M$ is a prime ideal.

Finally we claim that $P$ is an annihilator ideal. It is enough to show that $P$ contains no dense element. If not, let $x \in P \cap D(L)$. Then $x \notin M$ and by the maximality of $M$ with $I$, we have $(M \vee \{x\}) \cap I = \emptyset$. Let us consider $y \in (M \vee \{x\}) \cap I$. Then $y > t \wedge x$ for some $t \in M$. 


Since $y \in I$, therefore $t \wedge x \in I \subseteq P$. Thus by Lemma 2.2 $(t \wedge x)^{\perp} = t^{\perp} \cap x^{\perp} \subseteq I^{\perp} = I$, as $I$ is an annihilator ideal. Since $x$ is a dense element $x^{\perp} = L$, therefore $t^{\perp} \subseteq I$. Since $t \in t^{\perp}$, therefore $t \in I$. This contradicts the fact that $M \cap I = \emptyset$. Therefore $P \cap D(L) = \emptyset$ and hence by Theorem 4.3, $P$ is the required prime annihilator ideal.

References

(2010), 63-74.
[6] Y. S. Pawar and D. N. Mane, $\alpha$-ideals in 0-distributive semilattices and 0-

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