ANNIHILATOR IDEALS IN 0-DISTRIBUTIVE LATTICES

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Abstract. In this paper we generalized some results of annihilator ideals for 0-distributive lattices. We prove that the set of all annihilator ideals of a 0-distributive lattice form a Boolean algebra. We also prove Stone type separation theorem for annihilator ideals of 0-distributive lattices.

Key words and phrases. 0-distributive lattice, annihilator, ideal, Boolean algebra.

1. Introduction

A lattice L with 0 is called 0-distributive if for any $a, b, c \in L$ such that $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$. Varlet [7] introduced the notion of 0-distributive lattice for generalization of distributive lattices. For the background of lattice theory we refer the reader to the foundation monograph [3].

Annihilator ideals play an important role to study lattice theory. Cornish [2] and Davey [1] studied annihilator ideals for distributive lattices. In this paper we study annihilator ideals of a 0-distributive lattice. Here we generalize some results for 0-distributive lattices.

In Section 2, we prove some identities for annihilators which we need in this paper.

Pawar and Khopade [5] have studied α-ideals and annihilator ideals of a 0-distributive lattice as a consequence study of [6, 4]. They have mentioned that the set of all annihilator ideals of a 0-distributive lattice form a Boolean algebra. The lattice of all annihilator ideals is a sublattice of the lattice of all α-ideals of a 0-distributive lattice. In Section 3, we discuss annihilator ideals of a 0-distributive lattice. We prove that the set of all annihilator ideals of a 0-distributive lattice

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form a Boolean algebra. Our definition of supremum of two annihilator ideals is different from the definition of supremum of two α-ideals given in [5].

In Section 4, we found that prime ideals and annihilator ideals of a 0-distributive lattice are independent. In this section we show that a non-dense prime ideal of a 0-distributive semilattice is an annihilator ideal. We also give a characterization of prime ideal in 0-distributive lattices to be an annihilator ideals. We establish a sufficient condition for an annihilator ideal to be a prime ideal. We prove Stone type separation theorem for annihilator ideals of 0-distributive lattices.

2. Annihilators

Let L be a 0-distributive lattice and A be a non-empty subset of L. Define

$$A^{\perp} = \{x \in S \mid x \wedge a = 0 \text{ for all } a \in A\}.$$

Then $A^{\perp}$ is called the annihilator of A. If $a \in A$, then the annihilator of $\{a\}$ is denoted by $a^{\perp}$ and defined as

$$a^{\perp} = \{x \in L \mid x \wedge a = 0\}.$$

It is called the annulet generated by a.

The following identities are easy to prove and will be used throughout this paper.

Lemma 2.1. Let L be a lattice with 0 and $a, b \in L$. Then

(i) $a \in a^{\perp\perp}$.
(ii) If $a \leqslant b$, then $b^{\perp} \subseteq a^{\perp}$ and $a^{\perp\perp} \subseteq b^{\perp\perp}$.
(iii) $a^{\perp\perp\perp} = a^{\perp}$.
(iv) $a^{\perp} \cap a^{\perp\perp} = \{0\}$.

Now we have the following useful result.

Lemma 2.2. Let L be a 0-distributive lattice and $a, b \in L$. Then $(a \wedge b)^{\perp\perp} = a^{\perp\perp} \cap b^{\perp\perp}$.

Proof. Let $x \in a^{\perp\perp} \cap b^{\perp\perp}$ and $y \in (a \wedge b)^{\perp}$. Then $y \wedge a \wedge b = 0$. This implies $y \wedge a \in b^{\perp}$. Since $x \in b^{\perp\perp}$, therefore $x \wedge y \wedge a = 0$. This implies $x \wedge y \in a^{\perp}$. Since $x \in a^{\perp\perp}$, therefore $x \wedge y \in a^{\perp\perp}$. Thus $x \wedge y \in a^{\perp} \cap a^{\perp\perp} = 0$. This implies $x \wedge z = 0$ for all $z \in (a \wedge b)^{\perp}$. Then $x \in (a \wedge b)^{\perp\perp}$ and hence $a^{\perp\perp} \cap b^{\perp\perp} \subseteq (a \wedge b)^{\perp\perp}$. Converse is due to Lemma 2.1 (ii). Hence $(a \wedge b)^{\perp\perp} = a^{\perp\perp} \cap b^{\perp\perp}$.

3. Annihilator Ideals

Let L be a 0-distributive lattice. A non-empty subset F of L is called a filter if (i) $a \in F$ and $b \in L$ with $b > a$ implies $b \in F$, and (ii) $a, b \in F$ implies $a \wedge b \in F$. For any $a \in L$, the set

$$[a) = \{x \in S \mid a \leqslant x\}$$

is a filter called the principal filter generated by a.

A non-empty subset $\mathbf{I}$ of L is called an ideal if (i) $a \in \mathbf{I}$ and $b \in S$ with $b \leqslant a$ implies $b \in \mathbf{I}$, and (ii) $a, b \in \mathbf{I}$ implies $a \vee b \in \mathbf{I}$. For any $a \in L$ the set

$$(a] = \{x \in S \mid x \leqslant a\}$$

is an ideal called the principal ideal generated by a. The set of all ideals of L is denoted by $\mathbf{I}(L)$. It is well known that $\langle \mathbf{I}(L), \subseteq \rangle$ form a lattice as an ordered set. For $\mathbf{I}, \mathbf{J} \in \mathbf{I}(L)$, we denote $\mathbf{I} \wedge \mathbf{J} = \inf_{\mathbf{I}(L)}\{\mathbf{I}, \mathbf{J}\}$ and $\mathbf{I} \vee \mathbf{J} = \sup_{\mathbf{I}(L)}\{\mathbf{I}, \mathbf{J}\}$

Now we have the following useful properties for $\mathbf{I}(L)$.

Lemma 3.1. Let L be a lattice with 0 and $\mathbf{I}, \mathbf{J} \in \mathbf{I}(L)$. Then

(i) $\mathbf{I} \wedge \mathbf{J} = (0]$ if and only if $\mathbf{J} \subseteq \mathbf{I}^{\perp}$.
(ii) $\mathbf{I} \wedge \mathbf{I}^{\perp} = (0]$.
(iii) $\mathbf{I} \subseteq \mathbf{I}^{\perp\perp}$.
(iv) If $\mathbf{I} \subseteq \mathbf{J}$, then $\mathbf{J}^{\perp} \subseteq \mathbf{I}^{\perp}$.
(v) $\mathbf{I}^{\perp\perp\perp} = \mathbf{I}^{\perp}$.

Proof. (i) Let $b \in \mathbf{J}$. Then $a \wedge b = 0$ for all $a \in \mathbf{I}$, as $\mathbf{I} \wedge \mathbf{J} = (0]$. Thus $b \in \mathbf{I}^{\perp}$. Hence $\mathbf{J} \subseteq \mathbf{I}^{\perp}$. Converse is obvious.

(ii) Putting $\mathbf{J} = \mathbf{I}^{\perp}$ in (i).

(iii) Using (i) and (ii).

(iv) Let $x \in \mathbf{J}^{\perp}$. Then $x \wedge j = 0$ for all $j \in \mathbf{J}$. Since $\mathbf{I} \subseteq \mathbf{J}$, therefore $x \wedge i = 0$ for all $i \in \mathbf{I}$. Hence $\mathbf{J}^{\perp} \subseteq \mathbf{I}^{\perp}$.

(v) By (iii), $\mathbf{I}^{\perp} \subseteq \mathbf{I}^{\perp\perp\perp}$. Converse is true by (iii) and (iv). Hence $\mathbf{I}^{\perp\perp\perp} = \mathbf{I}^{\perp}$.

Now we have the following useful result for ideals.

Lemma 3.2. Let L be a 0-distributive lattice and $\mathbf{I}, \mathbf{J} \in \mathbf{I}(L)$. Then $(\mathbf{I} \wedge \mathbf{J})^{\perp\perp} = \mathbf{I}^{\perp\perp} \wedge \mathbf{J}^{\perp\perp}$.

Proof. Let $x \in \mathbf{I}^{\perp\perp} \wedge \mathbf{J}^{\perp\perp}$ and $y \in (\mathbf{I} \wedge \mathbf{J})^{\perp}$. Again let $i \in \mathbf{I}$ and $j \in \mathbf{J}$. Since $i \wedge j \in \mathbf{I} \wedge \mathbf{J}$ and $y \in (\mathbf{I} \wedge \mathbf{J})^{\perp}$, therefore $(y \wedge i) \wedge j = 0$. This implies $y \wedge i \in j^{\perp}$ for all $j \in J$. Hence $y \wedge i \in J^{\perp}$. Since $x \in J^{\perp\perp}$, we get $(x \wedge y) \wedge i = 0$ for all $i \in I$. Hence $x \wedge y \in I^{\perp}$. Since S is 0-distributive, therefore $I^{\perp\perp} \in \mathbf{I}(S)$ and hence $x \in I^{\perp\perp}$ implies $x \wedge y \in I^{\perp\perp}$. Thus $x \wedge y \in I^{\perp} \wedge I^{\perp\perp} = (0]$. Hence $x \wedge z = 0$ for all $z \in (I \wedge J)^{\perp}$. Therefore $x \in (I \wedge J)^{\perp\perp}$. Thus $I^{\perp\perp} \wedge J^{\perp\perp} \subseteq (I \wedge J)^{\perp\perp}$. Converse follows from Lemma 3.1(iv). Hence $(I \wedge J)^{\perp\perp} = I^{\perp\perp} \wedge J^{\perp\perp}$.

Lemma 3.3. Let L be a 0-distributive lattice and $I, J \in \mathbf{I}(L)$. Then

$$(I \vee J)^{\perp} = I^{\perp} \wedge J^{\perp}$$

Proof. Since $I, J \subseteq I \vee J$, so we have $(I \vee J)^{\perp} \subseteq I^{\perp} \wedge J^{\perp}$. On the other hand let $x \in I^{\perp} \wedge J^{\perp}$. Then $x \wedge t = 0$ for all $t \in I$ and $x \wedge s = 0$ for all $s \in J$. By the property of 0-distributivity of L, we have $x \wedge (t \vee s) = 0$. This implies $x \wedge i = 0$ for all $i \in I \vee J$. Hence $x \in (I \vee J)^{\perp}$. Thus $(I \vee J)^{\perp} = I^{\perp} \wedge J^{\perp}$.

Let L be a 0-distributive lattice. Then $A^{\perp}$ is an ideal of L for any non-empty subset A of L. An ideal I of L is called an annihilator ideal if $I = A^{\perp}$ for some non-empty subset A of L or equivalently if $I = I^{\perp\perp}$. The set of all annihilator ideals of L is denoted by $A(L)$. If $I, J \in A(L)$, then By Lemma 3.2, $I \wedge J \in A(L)$. Observe that $I \vee J$ may not be an annihilator ideal. For counterexample, consider the lattice $\mathbf{M}$ in Figure 1. Then $I = (a]$ and $J = (b]$ are annihilator ideals but $I \vee J = (c]$ is not an annihilator ideal. Thus $\langle A(L), \subseteq\rangle$ is not a sublattice of $\langle I(L), \subseteq\rangle$. Now we show that $\langle A(L), \subseteq\rangle$ is itself a lattice as an ordered set. For any $I, J \in A(L)$ we denote $I \sqcap J = \inf_{A(L)}\{I, J\}$ and $I \sqcup J = \sup_{A(L)}\{I, J\}$.

Indeed, we have the following result.

Theorem 3.4. Let L be a 0-distributive lattice. For any $I, J \in A(L)$ we have

(a) $I \sqcap J = I \wedge J$;
(b) $I \sqcup J = (I^{\perp} \wedge J^{\perp})^{\perp}$.

Proof. (a) Obvious.

(b) Clearly $I^{\perp} \wedge J^{\perp} \subseteq I^{\perp}, J^{\perp}$. This implies $I^{\perp\perp} = I \subseteq (I^{\perp} \wedge J^{\perp})^{\perp}$ and $J^{\perp\perp} = J \subseteq (I^{\perp} \wedge J^{\perp})^{\perp}$. Hence $I \sqcup J \subseteq (I^{\perp} \wedge J^{\perp})^{\perp}$. Now let K be an annihilator ideal such that $I, J \subseteq K$. Then $K^{\perp} \subseteq I^{\perp} \wedge J^{\perp}$. Since K is an annihilator ideal, this implies $(I^{\perp} \wedge J^{\perp})^{\perp} \subseteq K^{\perp\perp} = K$. Hence $I \sqcup J = (I^{\perp} \wedge J^{\perp})^{\perp}$.

Thus hA(L), $\mathbf{t}$, $\mathbf{ui}$ forms a lattice.

Lemma 3.5. Let L be a 0-distributive lattice and $I, J, K \in A(L)$. Then $(I \mathbf{t} J)$ u $K \subseteq I \mathbf{t} (J$ u $K)$.

Proof. We have

$$I \text{ u } K \text{ u } [I^{\perp} \text{ u } (J \text{ u } K)^{\perp}] = (0) \text{ and } J \text{ u } K \text{ u } [I^{\perp} \text{ u } (J \text{ u } K)^{\perp}] = (0)$$

$\Rightarrow K \text{ u } I^{\perp} \text{ u } (J \text{ u } K)^{\perp} \subseteq I^{\perp}$ and $K \text{ u } I^{\perp} \text{ u } (J \text{ u } K)^{\perp} \subseteq J^{\perp}$

$\Rightarrow K \text{ u } I^{\perp} \text{ u } (J \text{ u } K)^{\perp} \subseteq I^{\perp} \text{ u } J^{\perp}$

$\Rightarrow [K \text{ u } I^{\perp} \text{ u } (J \text{ u } K)^{\perp}] \text{ u } (I^{\perp} \text{ u } J^{\perp})^{\perp} = (0)$ by Lemma 3.1(i)

$\Rightarrow I^{\perp} \text{ u } (J \text{ u } K)^{\perp} \text{ u } [K \text{ u } (I^{\perp} \text{ u } J^{\perp})^{\perp}] = (0)$

$\Rightarrow K \text{ u } (I^{\perp} \text{ u } J^{\perp})^{\perp} \subseteq [I^{\perp} \text{ u } (J \text{ u } K)^{\perp}]^{\perp}$ by Lemma 3.1(i)

Hence by Theorem 3.4, we have $(I \mathbf{t} J) \text{ u } K \subseteq I \mathbf{t} (J \text{ u } K)$.

The following theorem shows that hA(L), $\mathbf{t}$, $\mathbf{ui}$ is a distributive lattice.

Theorem 3.6. Let L be a 0-distributive lattice. Then hA(L), $\mathbf{t}$, ui is a distributive lattice.

Proof. For any ideals $I, J, K \in A(L)$,

$$\begin{aligned}(I \mathbf{t} J) \text{ u } (I \mathbf{t} K) &\subseteq I \mathbf{t} [J \text{ u } (I \mathbf{t} K)] && \text{by Lemma 3.5}\\ &\subseteq I \mathbf{t} I \mathbf{t} (J \text{ u } K) && \text{by Lemma 3.5}\\ &= I \mathbf{t} (J \text{ u } K)\end{aligned}$$

The reverse inclusion is trivial and hence hA(L), $\mathbf{t}$, $\mathbf{ui}$ is a distributive lattice.

Clearly, $0^{\perp} = L$ and $L^{\perp} = (0)$. Now we have the following result.

Theorem 3.7. Let L be a 0-distributive lattice. Then hA(L); $\mathbf{t}$, u, $^{\perp}$, (0], L$\mathbf{i}$ is a Boolean Algebra.

Proof. We have hA(L); $\mathbf{t}$, u, (0], L$\mathbf{i}$ is a bounded distributive lattice. Now for any $I \in A(L)$, we have

$$I \text{ u } I^{\perp} = (0] \text{ and } I \mathbf{t} I^{\perp} = (I^{\perp} \text{ u } I^{\perp\perp})^{\perp} = (0]^{\perp} = L$$

Hence hA(L); $\mathbf{t}$, u, $^{\perp}$, (0], L$\mathbf{i}$ is a Boolean Algebra.

4. Separation Theorem for Annihilator Ideals

A prime ideal P is a proper ideal of L such that $a \wedge b \in P$ implies either $a \in P$ or $b \in P$. We denote the set of all prime ideals of L by $P_I(L)$. A prime ideal P is called minimal, if for any prime ideal $Q \subseteq P$ implies $P = Q$. A filter F of L is called a prime filter if $F = L$ and $L \setminus F$ is a prime ideal. It is well known that a filter F is a maximal filter if and only if $L \setminus F$ is a minimal prime ideal.

We observed in $\mathbf{M}$ (see Figure 1) that $I = (0]$ is a annihilator ideal but not prime. On the other hand $I = (c]$ is a prime ideal but not annihilator ideal. Thus the prime ideals and annihilator ideals of a 0-distributive lattice are independent.

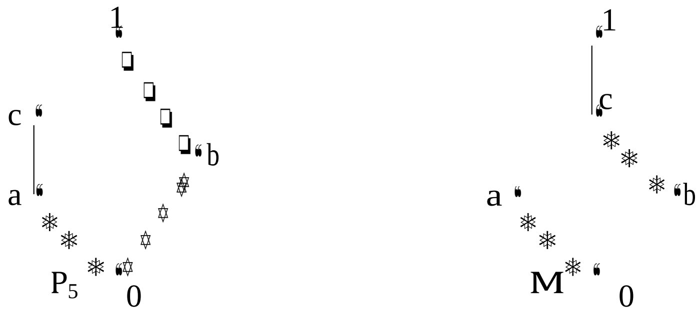

Figure 1. Two important 0-distributive semilattices

Let L be a 0-distributive lattice. An element $d \in L$ is called dense if $d^{\perp} = (0]$. The set of all dense element of L is denoted by $D(L)$. Thus

$$D(L) = \{x \in S \mid x^{\perp} = (0]\}.$$

It is well known that $D(L)$ is a filter. An ideal I of L is called a dense ideal if $I^{\perp} = (0]$.

Lemma 4.1. Let L be a 0-distributive lattice. Then a proper annihilator ideal contains no dense element.

Proof. Let I be a proper annihilator ideal of L and $x \in I \cap D(S)$. Then $x^{\perp} = (0]$. This implies $I^{\perp} = (0]$. Therefore I is a dense ideal and $I = I^{\perp\perp} = L$, which contradicts the fact that I is proper. Therefore $I \cap D(S) = \emptyset$.

Theorem 4.2. If a prime ideal P of a 0-distributive lattice L is non-dense, then P is an annihilator ideal.

Proof. Let P be a prime ideal of L. If P is non-dense, then $P^{\perp} = (0]$. This implies there exists $0 = x \in S$ such that $x \in P^{\perp}$. Hence $P^{\perp\perp} \subseteq x^{\perp}$ and so $P \subseteq x^{\perp}$. We show that $P = x^{\perp}$. If not, then let $a \in x^{\perp} \setminus P$. This implies $a \wedge x = 0 \in P$. Thus $x \in P$ as P is a prime ideal. This shows that $x \in P \cap P^{\perp} = (0]$, which is a contradiction. Hence $x^{\perp} = P$. Therefore P is an annihilator ideal.

The above results give us the following corollary.

Corollary 4.3. A prime ideal P of a 0-distributive lattice L is an annihilator ideal if and only if $P \cap D(L) = \emptyset$.

Now we have the following Theorem.

Theorem 4.4 (Separation Theorem for Annihilator Ideals). Let L be 0-distributive lattice, I be an annihilator ideal and F be a filter of L such that $I \cap F = \emptyset$. Then there is a prime annihilator ideal containing I and disjoint from F.

Proof. Let I be an annihilator ideal and F be a filter of a lattice L such that $I \cap F = \emptyset$. Define

$$X = \{G \mid G \text{ is a filter of } L \text{ such that } F \subseteq G \text{ and } I \cap G = \emptyset\}.$$

Clearly, $F \in X$ and X satisfies all the conditions of Zorn's Lemma. Thus X has a maximal element, say M. We show that M is a prime filter.

Suppose M is not prime. Then there are $a, b \in L \setminus M = P$ such that $a \vee b \in M$. By the maximality of M in X, we have $(M \vee [a)) \cap I = \emptyset$ and $(M \vee [b)) \cap I = \emptyset$. Let $x \in (M \vee [a)) \cap I = \emptyset$ and $y \in (M \vee [b)) \cap I = \emptyset$. Then $x, y \in I$ such that $x > m \wedge a$ and $y > n \wedge b$ for some $m, n \in M$. Since I is an annihilator ideal, $m \wedge a \in I = I^{\perp\perp}$ and $n \wedge b \in I = I^{\perp\perp}$. This implies $m \wedge a \wedge i = 0$ and $n \wedge b \wedge i = 0$ for all $i \in I^{\perp}$. Hence $(m \wedge n \wedge i) \wedge a = 0$ and $(m \wedge n \wedge i) \wedge b = 0$. Now since L is 0-distributive, we have $(m \wedge n \wedge i) \wedge (a \vee b) = 0$ for all $i \in I^{\perp}$. Thus $(m \wedge n) \wedge (a \vee b) \in I^{\perp\perp} = I$. Which is a contradiction to the fact that $I \cap M = \emptyset$. Therefore, M is a prime filter and hence $P = L \setminus M$ is a prime ideal.

Finally we claim that P is an annihilator ideal. It is enough to show that P contains no dense element. If not, let $x \in P \cap D(L)$. Then $x \notin M$ and by the maximality of M with I, we have $(M \vee [x)) \cap I = \emptyset$. Let us consider $y \in (M \vee [x)) \cap I$. Then $y > t \wedge x$ for some $t \in M$.

Since $y \in I$, therefore $t \wedge x \in I \subseteq P$. Thus by Lemma 2.2 $(t \wedge x)^{\perp\perp} = t^{\perp\perp} \cap x^{\perp\perp} \subseteq I^{\perp\perp} = I$, as I is an annihilator ideal. Since x is a dense element $x^{\perp\perp} = L$, therefore $t^{\perp\perp} \subseteq I$. Since $t \in t^{\perp\perp}$, therefore $t \in I$. This contradicts the fact that $M \cap I = \emptyset$. Therefore $P \cap D(L) = \emptyset$ and hence by Theorem 4.3, P is the required prime annihilator ideal.

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