Localized states of modified Dirac equation

S.B. Faruque[1†], S.D. Shuvo[1], M. Moniruzzaman[2] and P.K. Das[1]

[1]Department of Physics, Shahjalal University of Science and Technology, Sylhet 3114, Bangladesh.

[2]Department of Physics, Mawlana Bhashani Science and Technology University, Tangail 1902, Bangladesh.

†Corresponding author. Email: awsbf62@yahoo.com

Abstract

In this paper, we introduce an extension of the Dirac equation, very similar to Dirac oscillator, that gives stationary localized wave packets as eigenstates of the equation. The extension to the Dirac equation is achieved through the replacement of the momentum operator by a PT-symmetric generalized momentum operator. In the 1D case, the solutions represent bound particles carrying spin having continuous energy spectrum, where the envelope parameter defines the width of the packet without affecting the dispersion relation of the original Dirac equation. In the 2D case, the solutions are localized wave packets and are eigenstates of the third component of total angular momentum and involve Bessel functions of integral order. In the 3D case, the solutions are localized spherical wave packets with definite total angular momentum.

Keywords: Dirac equation, modified Dirac equation, energy spectrum, dispersion relation.

1 Introduction

Ordinarily, in quantum mechanics, the free particle solutions of the Dirac equation are plane waves with infinite uncertainty in position. But, infinite wave trains are not suitable for application. One, therefore, creates wave packets by superposing many quantum eigenstates. Wave packets are packets of wave function having finite width in position and in momentum, and as such, suitable for application. The aim of this paper is to present a particular coupling of the momentum of a Dirac particle with a position dependent dynamical operator which creates eigenstates of the equation that are localized stationary wave packets. This presents within the premise of Dirac theory an alternative way to the conventional creation of wave packets by superposition of many eigenstates; the generalized momentum operator does the job of wave packing. This process is similar to the process of nonlinear coupling between Coulomb motion of Rydberg electron and linearly polarized microwave field that generate electronic wave packets as stationary eigenstates of Schrodinger like equation [1-3].

Study of wave packets in the context of Dirac theory is itself an important task because of their use in nanophysics [4-7]. Moreover, relativistic wave packet pose a challenge to theory as such, many authors addressed this problem [8-11]. Localized stationary wave packets in orbit of atoms is an interesting topic being studied for long (see, for example, [12] and the references therein). Such stationary wave packets have numerous potential applications [12], such as in information processing, in cavity quantum electrodynamics and in precision spectroscopy etc. The present study elevates the issue to the relativistic regime where stationary wave localized packets are automatic and stable products of relativistic Dirac equation. And as such, our study opens up a new door to applications of relativistic quantum states. Experimental realization of such stationary wave packets can be anticipated as its predecessor Dirac oscillator has already been realized in experiments [13]. The present work is connected with the Dirac oscillator in the way that the Hamiltonian we employ here is derived from Dirac oscillator [14-16]. The solutions to Dirac oscillator are harmonic oscillator states; whereas, here we get qualitatively very similar states, namely, wave packet states. We present the equation and its properties in Section

2. In Section 3, we present the solutions in (1+1) freedom and discuss some of their properties. In Section 4, we present the solution in (2+1) freedom assuming the mass to be zero. In Section 5, we present the solution in (3+1) freedom. Finally, in Section 6, we summarize our work.

2 Dirac equation with PT-symmetric generalized momentum

The equation that gives wave packets as eigenstates is derived from Dirac oscillator [14] suppressing the Dirac matrix β in the coupling operator, i.e. in the free particle Dirac equation we replace $\vec{p}$ by $(\vec{p} - iq\vec{r})$ to obtain the following equation:

$$[c\vec{\alpha}.(\vec{p} - iq\vec{r}) + \beta mc^2]\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{1}$$

where, q is the envelope parameter that determines the width of the resulting wave packet, $\vec{\alpha}$, β are Dirac matrices, $\vec{p}$ and $\vec{r}$ are respectively momentum and coordinate of the fermion and c is the speed of light in free space. This equation differs from Dirac oscillator only in the matrix β which is present in the second factor within parentheses of the first term on the left side of Eq. (1) in case of Dirac oscillator. The operator $(\vec{p} - iq\vec{r})$ is PT-symmetric and can be easily checked. Under P (parity) transformation: $\vec{p} \to -\vec{p}$, $\vec{r} \to -\vec{r}$ and under T (time) reflection: $\vec{p} \to -\vec{p}$, $\vec{r} \to -\vec{r}$, $i \to -i$. Moreover, Eq. (1) can be generated from the Dirac Hamiltonian H_0 by the similarity transformation SH_0H^{-1} with, $S = \exp(\frac{-qr^2}{2\hbar})$. Hence, Eq. (1) changes the description of free Dirac particles from unlocalized states to localized states. We proceed from the next section to study its solutions. We reduce the equation in a way to be studied in (1+1) freedom in the next section and find the solutions. In passing, we discuss a little about why Eq. (1) is important. Similarity transformation and the concept of self-similarity are important foundations of fractals and iterated function systems. The determinant of the similarity transformation of a matrix is equal to the determinant of the original matrix. $[BAB^{-1}] = [B]AB^{-1} = [B]A[\frac{1}{[B]}] = [A]$. In this work, we explore the similarity between a plane wave moving in space as a solid particle and a wave packet that is also moving in space as particle. No doubt, the wave packet description is suitable and more worthy for application.

To create these wave packets, we write the original Dirac Hamiltonian as H_0 and consider the similarity transformation SH_0H^{-1} with, $S = \exp(\frac{-qr^2}{2\hbar})$. The states transform to $\Psi = \exp(\frac{-qr^2}{2\hbar})\Psi_0$, where q is an envelope parameter. As is known, a dilation corresponds to an expansion plus a translation. Further, similarity transformation transform objects in space to similar objects. But we get wave packets instead of free wave trains. And wave packets are similar in nature to free particle material points. Thus, we get an equivalent description of Dirac solution from two Dirac equations, where the new one is called modified Dirac equation.

3 Localized states in one dimension

We assume stationary solutions of Eq. (1) in the form $\Psi = \psi(\vec{r})\exp(-iEt/\hbar)$.

Then, the equation becomes

$$[c\vec{\alpha}.(\vec{p} - iq\vec{r}) + \beta mc^2]\psi(\vec{r}) = E\psi(\vec{r}) \tag{2}$$

Solutions to this equation in (1+1) freedom will be worked out with the assumption that the motion of the particle is along the z-direction with momentum p. To realize this, we use α_z in place of $\vec{\alpha}$ and replace $\vec{r}$ by z to obtain the governing equation as,

$$(c\alpha_z p - iqc\alpha_z z + \beta mc^2)\psi(z) = E\psi(z) \tag{3}$$

To construct the solution of this equation, first we note that the operator on the left of this equation, the Hamiltonian H, commutes with the operator of z-component of spin, Σ_z, i.e., $[\Sigma_z, H] = 0$. Hence, our solution should be simultaneous eigenstates of energy and spin. So, we write the solution in the form

$$\psi(z) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{4}$$

Inserting this into Eq. (3), we find the following coupled equations:

$$(cp - iqcz)u_3 = (E - mc^2)u_1\,, \tag{5}$$

$$(cp - iqcz)u_1 = (E + mc^2)u_3\,, \tag{6}$$

$$-(cp - iqcz)u_4 = (E - mc^2)u_2\,, \tag{7}$$

$$-(cp - iqcz)u_2 = (E + mc^2)u_4\,. \tag{8}$$

Following traditional methods, we first assume $u_2 = u_4 = 0$ and find from Eqs. (5) and (6),

$$(cp - iqcz)(cp - iqcz)u_1 = (E^2 - m^2c^4)u_1\,, \tag{9}$$

and

$$(cp - iqcz)(cp - iqcz)u_3 = (E^2 - m^2c^4)u_3\,. \tag{10}$$

Therefore, we see that u_1 and u_3 satisfy the same equation and thus, will have the same structure. We can now take $u_1 = u_3 = 0$ and find for u_2 and u_4 the same governing equations as Eq. (9) or (10). Thus, we need to solve only one equation, say, Eq. (9). We obtain from Eq. (9),

$$(p^2 - iqpz - iqzp - q^2z^2)u_1 = (\frac{E^2}{c^2} - m^2c^2)u_1\,. \tag{11}$$

To solve Eq. (11), we use $p=-i\hbar\frac{\partial}{\partial z}$ and at the same time make the coordinate *z* dimensionless by defining a new coordinate $z' = \sqrt{\frac{q}{\hbar}}z$ and transform Eq. (11) accordingly. In what follows only z' occurs and for brevity, we drop the prime and continue to write *z*. Then, we find

$$\frac{d^2u_1}{dz^2}+2z\frac{du_1}{dz}+z^2u_1+K_1u_1=0\,, \tag{12}$$

where

$$K_1=\frac{E^2}{q\hbar c^2}-\frac{m^2c^2}{q\hbar}+1\,. \tag{13}$$

We now consider a solution of the form

$$u_1(z)=\phi(z)\exp(-\frac{1}{2}z^2)\,. \tag{14}$$

Substituting this in Eq. (12), we obtain for $\phi(z)$, the governing equation,

$$\frac{d^2\phi}{dz^2}+\alpha^2\phi=0\,, \tag{15}$$

which immediately gives,

$$\phi(z)=\exp(i\alpha z)\,, \tag{16}$$

where

$$\alpha^2=\frac{E^2}{q\hbar c^2}-\frac{m^2c^2}{q\hbar}\,. \tag{17}$$

Hence, the full solution for u_1 is,

$$u_1(z)=\exp(i\sqrt{\frac{q}{\hbar}}\alpha z)\exp(-\frac{1}{2}\frac{q}{\hbar}z^2)\,, \tag{18}$$

which is a localized wave packet with the first factor giving the oscillation in space and the second factor giving the envelope of the packet with q governing the width of the packet. The energy associated with the wave packet is found as,

$$E=\pm\sqrt{\alpha^2c^2\hbar q+m^2c^4}=\pm\sqrt{\hbar^2k^2c^2+m^2c^4}=\pm E_k\,, \tag{19}$$

where, $k = \sqrt{\frac{q}{\hbar}}\alpha$ is the wave number. Surprisingly, k is independent of q as can be seen from Eq. (17). Hence, the solution can be written more lucidly as,

$$u_1(z) = \exp(ikz)\exp(-\frac{1}{2}\frac{q}{\hbar}z^2)\,. \tag{20}$$

We now turn our attention to the spinor (4). It has the two independent forms for spin up and spin down as follows:

$$\Psi_{up}(z) = N\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\exp(ikz)\exp(-\frac{1}{2}\frac{q}{\hbar}z^2)\,, \tag{21}$$

$$\Psi_{down}(z) = N\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\exp(ikz)\exp(-\frac{1}{2}\frac{q}{\hbar}z^2)\,, \tag{22}$$

The normalization factor can be calculated by demanding $\int \Psi^{\dagger}\Psi dz = 1$, which gives $N = \sqrt{\frac{1}{2}\sqrt{\frac{q}{4\pi\hbar}}}$. The states (21)-(22) are each eigenstates of spin and energy, and represent Gaussian wave packets with minimum uncertainty product of position and momentum. The widths of the packets are governed by the envelope parameter q. These states have continuous energy spectrum but they are not representing freely moving particles, rather the particles are bound. As such, it is better to say that the particles are quasiparticles. In Fig. (1) we draw the top component of Eq. (21) for specific values of q and k and we see that this is clearly a stationary localized wave packet.

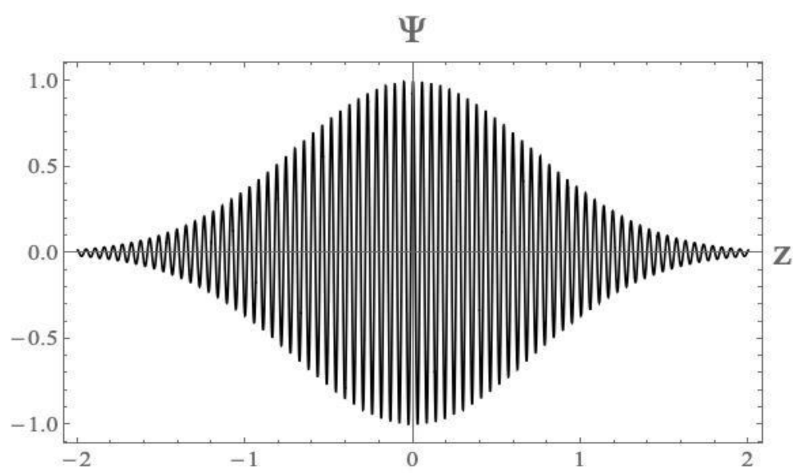

Figure 1: The unnormalized rendering of the top component of Eq. (21) where k=*116π* and $\frac{q}{2\hbar} = 1$. It is clear from the Figure that the solutions to Eq. (2) are localized wave packets.

4 Solution in two dimension for massless states

To solve Eq. (2) in 2D, we use *m=0*, for that gives the theory an opportunity to be applied to systems like graphene. And for that matter, we use in Eq. (2) $\vec{\sigma}$ matrices in place of $\vec{\alpha}$ matrices and assume $\vec{\sigma} = (\sigma_x, \sigma_y)$,

$\vec{p} = (p_x, p_y)$ and $\vec{r} = (x, y) = (r cos\varphi, r sin\varphi)$. The master equation can then be written as

$$\begin{pmatrix} 0 & cP_- \\ cP_+ & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{23}$$

where

$$P_+ = (p_x - iqx) + i(p_y - iqy), \tag{24}$$

$$P_- = (p_x - iqx) - i(p_y - iqy). \tag{25}$$

We transform Eq. (23) using polar coordinates as defined above and use the ansatz that the solutions are eigenstates of $J_z = L_z + \frac{\hbar}{2}\sigma_z$ with the eigenvalues of L_z being $m\hbar$. Hence, we write the solution as

$$\Psi(r,\varphi) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = e^{im\varphi}\begin{pmatrix} f(r) \\ e^{i\varphi} g(r) \end{pmatrix}. \tag{26}$$

Using the standard procedure, we obtain the second order differential equation satisfied by *f(r)* given by,

$$\frac{d^2 f}{dr^2} + (\frac{1}{r} + \frac{2q}{\hbar} r)\frac{df}{dr} + \frac{q^2}{\hbar^2} r^2 f - \frac{m^2}{r^2} f + K_2 f = 0, \tag{27}$$

where, $K_2 = \frac{E^2}{c^2\hbar^2} + \frac{2q}{\hbar}$. Now, an exactly similar equation is satisfied by *g(r)* with only $m \to m+1$. So, solving Eq. (27) only suffices for both the functions. Next, using an alternative coordinates as defined by $r' = \sqrt{\frac{q}{\hbar}} r$ and continuing with the use of r for r', we get from Eq. (27),

$$\frac{d^2 f}{dr^2} + (\frac{1}{r} + 2r)\frac{df}{dr} + (r^2 - \frac{m^2}{r^2} + K_3) f = 0, \tag{28}$$

where $K_3 = \frac{E^2}{\hbar q c^2} + 2$. Now, we write $f(r) = v(r) e^{-\frac{1}{2}r^2}$ and obtain for $v(r)$ the following equation:

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + (\rho^2 - \frac{m^2}{r^2}) v = 0, \tag{29}$$

where $\rho^2 = \dfrac{E^2}{\hbar q c^2}$. Solution of this equation are Bessel functions of integral order and one may choose any one from three types of Bessel functions of integral order. Here we choose the first kind of Bessel functions J_m and find the solution of Eq. (29) as $J_m(\rho\sqrt{\frac{q}{\hbar}}r)$. A similar calculation yields for the functions $g(r)$ the corresponding Bessel functions $J_{m+1}(\rho\sqrt{\frac{q}{\hbar}}r)$. Hence, we obtain the full solution as

$$\Psi(r,\varphi) = Ne^{im\varphi} e^{-\frac{1}{2}\frac{q}{\hbar}r^2} \begin{pmatrix} J_m(kr) \\ e^{i\varphi} J_{m+1}(kr) \end{pmatrix}, \tag{30}$$

where, $k = \rho\sqrt{\frac{q}{\hbar}} = \frac{E}{c\hbar}$ are the wave numbers. The dispersion relation is thus, $E = \pm\hbar kc$. Normalization constant N in Eq. (30) can be evaluated using the results of Ref. [17]. We obtain

$$N^2 \frac{\pi\hbar}{q} \exp(-\frac{\hbar k^2}{2q})[I_m(\frac{\hbar k^2}{2q}) + I_{m+1}(\frac{\hbar k^2}{2q})] = 1, \tag{31}$$

where, $I_m(z)$ and $I_{m+1}(z)$ are modified Bessel functions. The functions $\Psi(r,\varphi)$ of Eq. (30) are eigenstates of J_z, here the total angular momentum, and of energy. The solutions in the present case are wave packets as is evident from the structure of Eq. (30). Moreover, the solution (30) are similar in form as those found for graphene quantum dots in [18] except the appearance of the Gaussian factor in our case. Hence, we can assume that our extension of the Dirac equation affects only the extent of the wave function in space without affecting the energy level spectrum.

5 Solution in three dimension

We now solve Eq. (2) in full. To do so, we decompose $\Psi(\vec{r})$ as

$$\Psi(\vec{r}) = \begin{pmatrix} \psi_1(\vec{r}) \\ \psi_2(\vec{r}) \end{pmatrix}. \tag{32}$$

Using the standard representation of $\vec{\alpha}$ through the Pauli matrices $\vec{\sigma}$ and using Eq. (32) in Eq. (2), we obtain the two coupled equations given by

$$c\vec{\sigma}.\vec{p}\psi_2 - iqc\vec{\sigma}.\vec{r}\psi_2 = (E - mc^2)\psi_1, \tag{33}$$

$$c\vec{\sigma}.\vec{p}\psi_1 - iqc\vec{\sigma}.\vec{r}\psi_1 = (E - mc^2)\psi_2. \tag{34}$$

Based on the symmetries of the Dirac equation, we use the spin-angle functions [19] defined in two-component form as

$$y^{jm}_{j-\frac{1}{2}}(\hat{r}) = \begin{pmatrix} \sqrt{\dfrac{j+m}{2j}} Y_{j-\frac{1}{2} m-\frac{1}{2}} \\ \sqrt{\dfrac{j-m}{2j}} Y_{j-\frac{1}{2} m+\frac{1}{2}} \end{pmatrix}, \tag{35}$$

$$y^{jm}_{j+\frac{1}{2}}(\hat{r}) = \begin{pmatrix} -\sqrt{\dfrac{j-m+1}{2j+2}} Y_{j+\frac{1}{2} m-\frac{1}{2}} \\ \sqrt{\dfrac{j+m+1}{2j+2}} Y_{j+\frac{1}{2} m+\frac{1}{2}} \end{pmatrix}, \tag{36}$$

where Y's are spherical harmonics with *j* the total angular momentum quantum number and *m* being the magnetic quantum number associated with *j*. The functions given by Eqs. (35) and (36) are simultaneous eigenfunctions of L^2, S^2, J^2, J_z. Then, using standard procedure [19], we write

$$\psi_1(\vec{r}) = u(r) y^{jm}_{j-\frac{1}{2}}, \tag{37}$$

and

$$\psi_2(\vec{r}) = -iv(r) y^{jm}_{j+\frac{1}{2}}, \tag{38}$$

where in Eq. (38), the factor –*i* is included for later convenience. Now, we can write [19]

$$\vec{\sigma}.\vec{p} = (\vec{\sigma}.\hat{r})[-i\hbar \frac{\partial}{\partial r} + \frac{1}{r} i\vec{\sigma}.\vec{L}], \tag{39}$$

where $\vec{L}$ is the orbital angular momentum operator. Now,

$$(\vec{\sigma}.\vec{L}) y^{jm}_l = \kappa y^{jm}_l, \tag{40}$$

where $\kappa = -(\lambda+1)$ for $l = j + \dfrac{1}{2}$ and $\kappa = (\lambda - 1)$ for $l = j - \dfrac{1}{2}$, where $\lambda = j + \dfrac{1}{2}$. It is to be noted that [19]

$$\vec{\sigma}.\hat{r} y^{jm}_{l=j\pm\frac{1}{2}} = -y^{jm}_{l=j\mp\frac{1}{2}}. \tag{41}$$

Inserting Eq. (37)-(41) in Eq. (33) and rearranging, we obtain

$$(\frac{d}{dr} + \frac{\lambda+1}{r} + \frac{q}{\hbar} r) v(r) = (\frac{E - mc^2}{\hbar c}) u(r). \tag{42}$$

Similarly, we obtain from Eq. (34),

$$(\frac{d}{dr}-\frac{\lambda-1}{r}+\frac{q}{\hbar}r)u(r)=-(\frac{E+mc^2}{\hbar c})v(r)\,. \tag{43}$$

We can reduce Eq. (42) and (43) into uncoupled form by using simple algebra. We do this and use the dimensionless variable $r'=\sqrt{\frac{q}{\hbar}}r$ and find the following equations (where we keep on using *r* which is actually *r′*):

$$\frac{d^2u}{dr^2}+(\frac{2}{r}+2r)\frac{du}{dr}+(r^2-\frac{\lambda(\lambda-1)}{r^2})u+K_4u=0\,, \tag{44}$$

$$\frac{d^2v}{dr^2}+(\frac{2}{r}+2r)\frac{dv}{dr}+(r^2-\frac{\lambda(\lambda+1)}{r^2})v+K_4v=0\,, \tag{45}$$

where $K_4=\frac{E^2-m^2c^4}{\hbar qc^2}+3$. Using the assumption

$$u(r)=\xi(r)\exp(-\frac{1}{2}r^2)\,, \tag{46}$$

and using this in Eq. (44), we obtain

$$\frac{d^2\xi}{dr^2}+\frac{2}{r}\frac{d\xi}{dr}+(\gamma^2-\frac{\lambda(\lambda-1)}{r^2})\xi=0\,. \tag{47}$$

Similarly, using

$$v(r)=\chi(r)\exp(-\frac{1}{2}r^2)\,, \tag{48}$$

we obtain from Eq. (45)

$$\frac{d^2\chi}{dr^2}+\frac{2}{r}\frac{d\chi}{dr}+(\gamma^2-\frac{\lambda(\lambda+1)}{r^2})\chi=0\,, \tag{49}$$

where in Eqs. (47) and (49), $\gamma^2=\frac{E^2-m^2c^4}{\hbar qc^2}$. Solutions of Eq. (47) are spherical Bessel functions $j_{\lambda'}(\gamma r)$ and $n_{\lambda'}(\gamma r)$, where $\lambda'=\lambda-1$. Solutions of Eq. (49) are also spherical Bessel functions, namely $j_{\lambda}(\gamma r)$ and $n_{\lambda}(\gamma r)$. Hence, we can write explicitly, using only regular solutions and restoring the original variable *r*,

$$\psi_1(\vec{r})=Nj_{\lambda'}(kr)\exp(-\frac{1}{2}\frac{q}{\hbar}r^2)y_{\lambda'}^{jm}\,, \tag{50}$$

and

$$\psi_2(\vec{r}) = -iNj_{\lambda}(kr)\exp(-\frac{1}{2}\frac{q}{\hbar}r^2)y_{\lambda}^{jm}, \quad (51)$$

where

$$E^2 = \hbar^2 k^2 c^2 + m^2 c^4, \quad (52)$$

or,

$$k = \pm\frac{\sqrt{E^2 - m^2 c^4}}{\hbar c}. \quad (53)$$

Evidently, k is independent of q, the envelope parameter. Next, we normalize the wavefunction (32) with ψ_1 and ψ_2 given by Eqs. (50) and (51) where N is the normalization constant. Using results of Ref. [17], we obtain

$$N^2\left(\frac{\hbar}{q}\right)^{\frac{3}{2}}\frac{\sqrt{\pi}}{4}\exp\left(-\frac{\hbar k^2}{2q}\right)\left[f_{\lambda'}\left(\frac{\hbar k^2}{2q}\right)+f_{\lambda}\left(\frac{\hbar k^2}{2q}\right)\right]=1, \quad (54)$$

where $f_n(z) = \sqrt{\frac{\pi}{2z}}I_{n+\frac{1}{2}}(z)$ are the modified spherical Bessel function of the first kind. For completeness, we now write explicitly the spinor for $j = \frac{3}{2}$ and $m = \frac{3}{2}$, which is

$$\Psi^{\frac{3}{2}\frac{3}{2}} = N\exp\left(-\frac{1}{2}\frac{q}{\hbar}r^2\right)\begin{pmatrix} j_1(kr)Y_{11} \\ 0 \\ ij_2(kr)\sqrt{\frac{1}{5}}Y_{21} \\ -ij_2(kr)\sqrt{\frac{4}{5}}Y_{22} \end{pmatrix}. \quad (55)$$

The solutions found, namely Eq. (32) with ψ_1 and ψ_2 given by Eqs. (50) and (51) with the specific example given by Eq. (55) are wave packets in three dimensions carrying total angular momentum and its *z*-component given by the quantum numbers *j* and *m* as conserved quantities. Hence, we get here stationary spherical Bessel wave packets carrying angular momentum. Thus, we have found a complete picture of the solutions of Eq. (2) which is an extension of the Dirac equation very similar to Dirac oscillator.

6 Summary and Conclusion

In this paper, we have presented an extension of the Dirac equation, very similar to the Dirac oscillator, given by Eq. (1). We have solved the (1+1) case of the equation and the solutions are given by Eqs. (21)-(22). The solutions are spinor wave packets carrying definite spin ($^1/_2$ *or* $-\,^1/_2$) and continuous energy spectrum. They represent bound quasiparticles although the spectrum is continuous. The states are of minimum position-momentum uncertainty product, the width in position and momentum space are determined by the envelope parameter q. This parameter entering Eq. (1) via the operator $(-iq\vec{r})$ does not affect the dispersion relation, given by Eq. (19), but only packs the otherwise sinusoidal waves into a Gaussian envelope. This is why we call Eq. (1) the “wave packing Dirac equation”. The solutions can also be looked at as representing freely moving particles, but in that case they suffer dispersion owing to the nonlinear dispersion relation (19). Extension of the system to the massless case is easy and the solutions remain same. Finally, in section 5, we have solved the equation in full, using spherical polar coordinates and spin-angle functions. The same envelope function shows up and it envelopes the free particle functions given by spherical Bessel functions. In all the three cases, namely, one to three dimensions, we obtain wave packets carrying angular momentum. In the 1D case, it is the spin that is conserved; in the 2D case, it is the total angular momentum, which in this case is the *z*-component of total angular momentum, is conserved. In the 3D case, it is the total angular momentum and its *z*-component that are conserved quantities in the solution. And in all cases, the solutions are stationary wave packets. Our work will be very useful if we can use the wave packet states to represent interaction between Dirac particles, but that requires tremendous work which is beyond the scope of this paper. In conclusion, we have found a procedure to localize the otherwise unlocalized Dirac spinors by transforming the free particle Dirac Hamiltonian to a new form via a similarity transformation.

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