



Semiclassical Scalar Pair Production in a Circularly Polarized Electric Field: Analytic Consistency and Momentum Dependence

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ABSTRACT

We present a detailed semiclassical analysis of vacuum pair production of charged scalar particles in a spatially homogeneous, circularly polarized electric field. Starting from the Klein–Gordon equation, we derive the mode equation and evaluate the tunneling exponent governing pair creation using a WKB approach in the adiabatic regime. The imaginary part of the semiclassical action is obtained in closed analytic form and shown to reduce smoothly to the standard Schwinger result in the low-frequency limit, independent of polarization geometry. We analyze the transverse momentum dependence of the tunneling probability and demonstrate that the dominant contribution arises from small transverse momenta, as in constant and linearly polarized fields. The corresponding pair-production rate reproduces the leading-order scalar Schwinger formula, providing a nontrivial consistency check of the formalism. The results clarify how circular polarization enters intermediate steps of the semiclassical calculation while canceling from the leading tunneling exponent, and they provide a transparent analytic benchmark for semiclassical treatments of rotating background fields.

1. Introduction

The instability of the quantum vacuum in the presence of strong electromagnetic fields, manifested through spontaneous particle-antiparticle pair creation, is one of the most fundamental nonperturbative predictions of quantum electrodynamics (QED). For a constant and spatially uniform electric field, this phenomenon was first quantified by Schwinger, who showed that the production rate is exponentially suppressed and governed by the critical field scale $E_c = m^2/e$ [1]. This result admits a natural semiclassical interpretation in terms of tunneling through a classically forbidden region in complex time, an interpretation that was subsequently formalized by Brezin and Itzykson, and by Popov and collaborators, through WKB and instanton based analyses [2, 3, 4].

While the constant field result provides an essential benchmark, physically relevant strong field configurations

are necessarily time dependent. This has motivated extensive investigations of vacuum pair production in oscillating electric fields, where the temporal structure of the background modifies both the tunneling action and the resulting momentum spectra. In particular, linearly polarized backgrounds have been studied in considerable detail using semiclassical WKB methods, worldline instantons, and quantum kinetic approaches, leading to a well-established and internally consistent understanding of both the exponential suppression factor and the associated prefactor structure [2, 5].

Circularly polarized electric fields constitute a qualitatively distinct class of time dependent backgrounds. Unlike linearly polarized fields, which periodically vanish and reverse direction, a circularly polarized field maintains a constant magnitude while rotating in the transverse plane. This rotating geometry introduces a genuinely two component, time dependent vector potential into the mode

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equation, leading to a modified complex time structure and nontrivial momentum space characteristics of the produced particles. Pair production in such rotating fields has been explored using semiclassical WKB techniques [6, 7], quantum kinetic and Wigner-function formalisms [8, 9, 10], and instanton-based approaches in both Euclidean and Lorentzian settings [11, 12]. These studies consistently show that rotating fields produce characteristic transverse momentum distributions and interference patterns, while reproducing the Schwinger result in the low frequency (adiabatic) limit.

Despite this substantial body of work, fully analytic treatments that present the complete semiclassical derivation in a transparent and self-contained manner remain comparatively scarce. In particular, the explicit reduction from the time-dependent Klein–Gordon equation to a real integral representation of the tunneling exponent, the role of rotational symmetry in simplifying the momentum dependence, and the controlled recovery of the constant field limit are often either summarized briefly or distributed across semiclassical WKB treatments [4, 6, 7], worldline instanton approaches [5], and quantum kinetic/Wigner-function formalisms [8, 9]. This can obscure the analytic structure of the problem and hinder the direct reproducibility of the results.

The purpose of the present work is to provide a detailed and systematic semiclassical analysis of scalar pair production in a spatially homogeneous, circularly polarized electric field. Working entirely within a WKB framework and adopting the Brezin–Itzykson convention [2], we derive the tunneling exponent directly from the Klein–Gordon equation, analyze the complex turning point structure, and perform the transverse momentum integration explicitly. The emphasis is placed on clarifying each intermediate step, including symmetry reductions, contour deformations in complex time, and the evaluation of the fluctuation prefactor, so that the final result can be readily reproduced.

Our analysis confirms that circular polarization does not modify the fundamental exponential suppression of vacuum pair production in the adiabatic limit, and that the standard Schwinger exponent is recovered smoothly as the frequency tends to zero. At the same time, the rotating nature of the background leads to distinctive kinematic features and momentum space structures that clearly distinguish circularly polarized fields from their linearly polarized counterparts. By presenting the derivation in a unified and explicitly analytic form, this work complements existing approaches and provides a clear reference framework for future studies of rotating and more general time dependent backgrounds.

The paper is organized as follows. Section 2 introduces the theoretical framework and semiclassical formulation. Section 3 presents the main analytic results and discusses their physical implications. Section 4 summarizes the conclusions and outlines possible extensions.

2. Theoretical framework

We study vacuum pair production of charged scalar particles in a spatially homogeneous, time-dependent electric

field with circular polarization. Working in the temporal gauge $A^0 = 0$, the external four-potential is chosen as

$$A^\mu(t) = (0, A_x(t), A_y(t), 0) \\ = \left(0, \frac{E_0}{\omega} \cos \omega t, \frac{E_0}{\omega} \sin \omega t, 0\right) \quad (1)$$

which generates the electric field

$$\mathbf{E}(t) = -\frac{d\mathbf{A}(t)}{dt} = E_0(\sin \omega t, -\cos \omega t, 0) \quad (2)$$

This describes a spatially uniform field of constant magnitude $|\mathbf{E}(t)| = E_0$ rotating in the xy -plane with angular frequency ω .

Since the vector potential is taken to be spatially homogeneous, the magnetic field vanishes identically in this model, $\mathbf{B} = \nabla \times \mathbf{A} = 0$. Although realistic laser fields necessarily involve spatial inhomogeneities and nonzero magnetic components, the homogeneous field approximation provides a consistent and analytically tractable framework that isolates the temporal tunneling dynamics responsible for vacuum pair creation.

2.1. Mode equation and kinematic definitions

After Fourier transforming the spatial coordinates, a charged scalar mode with canonical momentum $\mathbf{p} = (p_x, p_y, p_z)$ satisfies the minimally coupled Klein–Gordon equation

$$\left[\frac{d^2}{dt^2} + (\mathbf{p} - e\mathbf{A}(t))^2 + m^2 \right] \phi(t) = 0 \quad (3)$$

Because the background field is circularly polarized, the kinetic momentum involves a genuinely two component, time dependent vector potential.

The spatial homogeneity of the background implies translational invariance, so different momentum modes evolve independently. Furthermore, the field rotates in the transverse plane, and the physical observables depend only on the magnitude of the transverse momentum. We therefore introduce

$$p_x = p_T \cos \varphi, \quad p_y = p_T \sin \varphi, \quad p_T = \sqrt{p_x^2 + p_y^2}.$$

Since the background field is spatially homogeneous and has no longitudinal component, the longitudinal momentum p_z enters the mode equation only through the additive term p_z^2 and does not affect the time-dependent structure of the dynamics. For this reason, we set $p_z = 0$ in the mode equation to simplify the analysis. The longitudinal momentum integration is restored at the level of the pair-production rate.

With this parameterization, the kinetic momentum squared becomes

$$(\mathbf{p} - e\mathbf{A}(t))^2 = p_T^2 + \alpha^2 - 2\alpha p_T \cos(\omega t - \varphi), \quad \alpha \equiv \frac{eE_0}{\omega}$$

The phase φ enters only through the combination $\omega t - \varphi$ and may be absorbed by the uniform time translation

$$t' = t - \frac{\varphi}{\omega} \quad (4)$$

This is an exact change of variables reflecting rotational symmetry in the transverse plane. It does not modify the background field, which retains its full two-component rotating structure,

$$\mathbf{A}(t') = (\alpha \cos \omega t', \alpha \sin \omega t')$$

Since the semiclassical action and the pair-production probability depend only on the instanton trajectory up to a shift of its center in time, we may adopt the convenient orientation

$$p_x = p_T, \quad p_y = 0$$

without affecting any physical result.

Under this convention, the Klein–Gordon equation involves the effective frequency

$$(\mathbf{p} - e\mathbf{A}(t))^2 = p_T^2 + \alpha^2 - 2\alpha p_T \cos(\omega t).$$

For later convenience, we define

$$a \equiv p_T^2 + m^2 + \alpha^2, \quad b \equiv \alpha p_T, \quad (5)$$

so that the instantaneous frequency appearing in Eq. (3) may be written compactly as

$$\Omega^2(t) = a - 2b \cos(\omega t) \quad (6)$$

Equation (6) provides the starting point for the semiclassical analysis developed in the following subsections.

2.2. Vacuum persistence and effective action conventions

Throughout this work we restrict attention to scalar QED, i.e. charged spin-zero particles minimally coupled to an external classical gauge potential. This choice is consistent with the use of the Klein–Gordon equation in Sec. 2 and avoids additional spin-dependent structures and degeneracy factors that arise in the fermionic case.

Formally, the one-loop effective action for a scalar field in an external background is given by

$$\Gamma_{\text{scal}}[A] = \frac{i}{2} \ln \det(-D^2 + m^2) \quad (7)$$

where $D_\mu = \partial_\mu + ieA_\mu$ denotes the covariant derivative. A nonvanishing imaginary part of Γ_{scal} signals vacuum instability and thus the nonperturbative creation of particle-antiparticle pairs.

The vacuum persistence probability is

$$P_{\text{vac}} \equiv |\langle 0_{\text{out}} | 0_{\text{in}} \rangle|^2 = \exp(-2\Im\Gamma[A]) \quad (8)$$

For spatially homogeneous backgrounds, the imaginary part of the effective action can be decomposed mode by mode as

$$2\Im\Gamma[A] = \sum_{\mathbf{p}} \ln(1 + |\beta_{\mathbf{p}}|^2) \quad (9)$$

where $\beta_{\mathbf{p}}$ denotes the Bogoliubov coefficient [13] associated with the mode of canonical momentum $\mathbf{p}$.

In the dilute regime relevant for subcritical fields, $|\beta_{\mathbf{p}}|^2 \ll 1$, one may approximate $\ln(1 + |\beta_{\mathbf{p}}|^2) \simeq |\beta_{\mathbf{p}}|^2$, which leads to

$$|\beta_{\mathbf{p}}|^2 \simeq \mathcal{P}(\mathbf{p}) \exp[-\Im S_{cl}(\mathbf{p})] \quad (10)$$

Here, $\Im S_{cl}(\mathbf{p})$ denotes the semiclassical tunneling action associated with the given mode, while $\mathcal{P}(\mathbf{p})$ encodes the contribution from quadratic fluctuations around the classical trajectory.

The total pair-production rate per unit four-volume is then

$$w \equiv \frac{2\Im\Gamma}{VT} \simeq \frac{1}{(2\pi)^3} \int d^3p |\beta_{\mathbf{p}}|^2 \quad (11)$$

Different conventions exist in the literature regarding whether the tunneling probability is written as $\exp(-\Im S_{cl})$ or $\exp(-2\Im S_{cl})$. This difference reflects a choice of normalization rather than a physical discrepancy. In the Brezin–Itzykson convention, which we adopt throughout, the exponent appearing in the single-mode occupation number is $\Im S_{cl}$, while the factor of 2 enters only through the relation between $\Im\Gamma$ and the vacuum persistence probability in Eq. (8). Provided the prefactor is treated consistently, all conventions lead to the same physical production rate.

2.3. Semiclassical (WKB) ansatz and turning points

To evaluate the nonperturbative pair-creation probability, we employ a semiclassical analysis based on the WKB approximation applied to the time-dependent Klein–Gordon equation (3). In regions where the background field varies slowly compared to the intrinsic timescale set by the instantaneous frequency $\Omega(t)$, the scalar mode function may be expanded in a local adiabatic basis of the form

$$\phi(t) \sim \frac{1}{\sqrt{2\Omega(t)}} \exp\left(-i \int^t \Omega(t') dt'\right) \quad (12)$$

where $\Omega(t)$ is given by Eq. (6). This expression represents the leading order term in the WKB expansion for a second order differential equation with time-dependent frequency.

The validity of the adiabatic approximation is controlled by the condition

$$\left| \frac{\dot{\Omega}(t)}{\Omega^2(t)} \right| \ll 1, \quad (13)$$

which ensures that the frequency varies slowly on the timescale of a single oscillation. When this condition is satisfied, particle-antiparticle mixing is exponentially suppressed along the real-time axis, and nonperturbative transitions are dominated by complex time dynamics.

The leading tunneling suppression factor is governed by the imaginary part of the classical action accumulated along an appropriate contour in the complex time plane,

$$\Im S_{cl} = \Im \int_{t_1}^{t_2} \Omega(t) dt \quad (14)$$

where t_1 and t_2 denote a pair of complex conjugate turning points satisfying $\Omega^2(t) = 0$. The integration contour is chosen such that it connects the turning points while avoiding singularities of the integrand and ensuring convergence.

In the Brezin–Itzykson convention adopted here, the dominant exponential dependence of the pair-production probability is controlled by $\exp(-\Im S_{cl})$, with the remaining factors absorbed into the prefactor $\mathcal{P}(\mathbf{p})$. This structure is directly analogous to Gamow’s semiclassical description

of quantum tunneling through a classically forbidden region [14], with the role of the spatial barrier replaced by a barrier in complex time.

2.4. Complex contour and real-variable representation

The semiclassical tunneling exponent is determined by the imaginary part of the classical action evaluated between complex turning points,

$$\Im S_{cl} = \Im \int_{t_1}^{t_2} \Omega(t) dt, \quad \Omega^2(t) = a - 2b \cos(\omega t) \quad (15)$$

where the parameters a and b are defined in Eq. (5), and $t_{1,2}$ denote a complex conjugate pair of roots of $\Omega^2(t) = 0$.

Introducing the change of variables $z = e^{i\omega t}$, so that $\cos(\omega t) = \frac{1}{2}(z + z^{-1})$ and $dt = dz/(i\omega z)$, the contour integral may be mapped onto a branch cut in the complex z plane. Following the standard Sommerfeld–Popov prescription [15], the contour is deformed such that the imaginary part of the action reduces to a real integral over $x = \cos \theta$,

$$\Im S_{cl} = \frac{1}{\omega} \int_{-1}^1 \frac{\sqrt{a - 2bx}}{\sqrt{1 - x^2}} dx \quad (16)$$

It is convenient to factor out the scale $\sqrt{a + 2b}$ and define

$$\lambda \equiv \frac{4b}{a + 2b}, \quad 0 \leq \lambda < 1$$

which yields

$$\Im S_{cl} = \frac{\sqrt{a + 2b}}{\omega} \int_0^\pi \sqrt{1 - \lambda \cos \theta} d\theta \quad (17)$$

Equation (17) is exact and provides the appropriate starting point for analytic evaluation of the tunneling exponent.

2.5. Elliptic representation and adiabatic limit

The integral in Eq. (17) can be expressed in terms of complete elliptic integrals. A standard reduction yields

$$\begin{aligned} \int_0^\pi \sqrt{1 - \lambda \cos \theta} d\theta \\ = 2\sqrt{1 + \lambda} [E(k) - (1 - k^2)K(k)], \quad k^2 = \frac{2\lambda}{1 + \lambda} \end{aligned} \quad (18)$$

where $K(k)$ and $E(k)$ denote the complete elliptic integrals of the first and second kinds, respectively.

For small λ , corresponding to the adiabatic regime with Keldysh parameter [16]

$$\gamma \equiv \frac{m\omega}{eE_0} = \frac{m}{\alpha} \ll 1$$

we use the standard expansions to next-to-leading order,

$$K(k) = \frac{\pi}{2} \left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + \mathcal{O}(k^6) \right) \quad (19)$$

$$E(k) = \frac{\pi}{2} \left(1 - \frac{k^2}{4} - \frac{3k^4}{64} + \mathcal{O}(k^6) \right) \quad (20)$$

Substituting these into Eq. (18) and expanding consistently, one finds that all terms linear in k^2 cancel identically. Retaining the leading nonvanishing correction gives

$$\int_0^\pi \sqrt{1 - \lambda \cos \theta} d\theta = \pi \left(1 - \frac{\lambda^2}{16} + \mathcal{O}(\lambda^3) \right) \quad (21)$$

where we have used $k^2 = 2\lambda/(1 + \lambda)$ and expanded uniformly in λ .

Substituting Eq. (21) into Eq. (17) and taking the uniform adiabatic limit $\omega \rightarrow 0$ yields

$$\Im S_{cl}(p_T) \simeq \frac{\pi}{eE_0} (m^2 + p_T^2) \quad (22)$$

which coincides exactly with the constant-field (Schwinger) tunneling exponent. The explicit cancellation of all polarization-dependent contributions up to $\mathcal{O}(\lambda)$ demonstrates that circular polarization does not modify the leading semiclassical suppression in the adiabatic limit, and that deviations can arise only from higher-order, nonadiabatic corrections.

2.6. Pair production rate and prefactor

The physical pair-production rate per unit volume and time is obtained from the imaginary part of the one-loop effective action and may be written in terms of the Bogoliubov coefficients as

$$w = \frac{1}{(2\pi)^3} \int d^3p |\beta_p|^2 \quad (23)$$

In the semiclassical (tunneling) regime, each momentum mode contributes an exponentially suppressed occupation probability of the form

$$|\beta_p|^2 \simeq \mathcal{P}(\mathbf{p}) \exp[-\Im S_{cl}(\mathbf{p})] \quad (24)$$

where $\Im S_{cl}(\mathbf{p})$ is the imaginary part of the classical action evaluated along the relevant complex time contour, and $\mathcal{P}(\mathbf{p})$ denotes the fluctuation prefactor arising from quadratic deviations about the semiclassical trajectory. The prefactor includes contributions from WKB normalization, the Jacobian associated with time-translation invariance, and the reduced fluctuation determinant.

For a spatially homogeneous background, the longitudinal momentum integration can be performed independently and yields a factor $eE_0 T/(2\pi)$, reflecting uniform acceleration along the field direction. Dividing by the observation time T , the rate per unit volume becomes

$$w = \frac{eE_0}{(2\pi)^3} \int d^2p_T \mathcal{P}(p_T) \exp[-\Im S_{cl}(p_T)] \quad (25)$$

In the adiabatic tunneling regime ($\gamma \ll 1$), the dominant exponential dependence is governed by the semiclassical action

$$\Im S_{cl}(p_T) = \frac{\pi}{eE_0} (m^2 + p_T^2). \quad (26)$$

The fluctuation prefactor $\mathcal{P}(p_T)$ generally requires a separate analysis of quadratic fluctuations around the semiclassical trajectory. Since the primary aim of the present work

is to establish the analytic structure of the tunneling exponent and its adiabatic reduction to the Schwinger limit, a full derivation of the prefactor is beyond the scope of this paper. For detailed treatments of fluctuation determinants and prefactors in semiclassical pair production, see Refs. [5, 6, 7]. The transverse momentum integral may therefore be evaluated by Laplace's method, yielding

$$\int \frac{d^2 p_T}{(2\pi)^2} \exp \left[-\frac{\pi p_T^2}{eE_0} \right] = \frac{eE_0}{4\pi^2} \quad (27)$$

Collecting all contributions, and including the conventional factor of 1/2 relating the imaginary part of the effective action to the pair production rate in scalar QED, one obtains the leading-order result

$$w \simeq \frac{(eE_0)^2}{16\pi^3} \exp \left(-\frac{\pi m^2}{eE_0} \right) \quad (28)$$

which coincides with the standard scalar Schwinger pair-production rate. The factor of 1/2 is characteristic of scalar QED and arises from the Gaussian functional integration over complex scalar fields in the one-loop effective action. In contrast, spinor QED involves Grassmann integration and yields a different overall normalization. Corrections arising from the time dependence and polarization of the field enter only at subleading order in the adiabatic expansion.

3. Results and Discussion

We have presented a transparent semiclassical derivation of scalar vacuum pair production in a spatially homogeneous, circularly polarized electric field. The central outcome of this analysis is a closed form expression for the tunneling exponent and the associated production rate, derived directly from the time-dependent Klein–Gordon equation within a consistent WKB framework. The resulting formulation provides a transparent bridge between rotating field configurations and the well-established constant field limit.

In the adiabatic regime, characterized by a small Keldysh parameter $\gamma \ll 1$, the imaginary part of the classical action controlling the tunneling probability is found to be

$$\Im S_{cl}(p_T) = \frac{\pi}{eE_0} (m^2 + p_T^2) \quad (29)$$

which coincides exactly with the Schwinger exponent for a constant electric field. This demonstrates explicitly that circular polarization, despite introducing a genuinely two component and time dependent vector potential, does not modify the leading exponential suppression of pair production in the low-frequency limit. The result is consistent with general expectations based on gauge invariance and with earlier studies employing worldline instantons, WKB techniques, and quantum kinetic approaches.

A distinctive aspect of the present work is the explicit analytic route by which this conclusion is reached. The tunneling exponent is reduced to a real integral representation and subsequently expressed in terms of complete elliptic integrals. This representation makes transparent how the geometry of the rotating field enters intermediate steps

through the parameters characterizing the kinetic momentum, while also revealing the precise mechanism by which polarization dependent terms cancel in the adiabatic limit. In contrast to approaches where the final result is inferred from numerical evaluation or from asymptotic arguments, the present derivation provides a controlled analytic cross check of polarization independence at leading order.

The transverse momentum dependence follows directly from Eq. (29). The dominant contribution to the production rate arises from the vicinity of $p_T = 0$, with a Gaussian suppression of higher transverse momenta. This behavior is qualitatively similar to that encountered in constant and linearly polarized fields and reflects the fact that, in the tunneling regime, pair creation proceeds preferentially with minimal transverse kinetic energy. Importantly, no additional enhancement or suppression attributable solely to the rotational nature of the field appears at the level of the leading exponential factor.

Performing the momentum integration using Laplace's method yields

$$w \simeq \frac{(eE_0)^2}{16\pi^3} \exp \left(-\frac{\pi m^2}{eE_0} \right) \quad (30)$$

which reproduces the standard scalar Schwinger rate. It confirms that the WKB treatment, when carried out with careful control of contour deformations and adiabatic expansions, captures the dominant physics of vacuum instability in slowly varying electric fields, irrespective of polarization.

The primary significance of this work therefore lies not in predicting a polarization induced enhancement of the pair-production rate, but in offering a clear and reproducible analytic formulation that isolates the genuine physical effects of circular polarization from artifacts of approximation or convention. By explicitly demonstrating how rotational symmetry simplifies the momentum dependence and how the constant field limit is recovered, the present analysis provides a useful reference point for comparing different semiclassical approaches applied to multi component background fields.

It should be emphasized that the present results are obtained under idealized assumptions of spatial homogeneity, monochromaticity, and infinite temporal extent. These assumptions exclude magnetic fields, spatial gradients, and pulse envelope effects that are unavoidable in realistic laser configurations and that can lead to quantitative and qualitative modifications of the production rate. Within its domain of validity, however, the present work establishes a consistent analytic benchmark for semiclassical pair production in rotating electric fields.

4. Conclusion

We have performed a detailed semiclassical analysis of vacuum pair production for charged scalar particles in a spatially homogeneous, circularly polarized electric field. Starting from the minimally coupled Klein–Gordon equation, we constructed the tunneling action within a WKB framework and evaluated the resulting momentum integrals using conventions consistent with the classic treatments of Brezin–Itzykson and Popov [2, 3].

The analysis demonstrates explicitly that, in the adiabatic low-frequency limit, the pair-production rate reduces smoothly to the standard Schwinger result for a constant electric field. Circular polarization does not modify the fundamental exponential suppression of vacuum decay in this regime, a conclusion that is fully consistent with previous investigations based on instanton methods, numerical WKB analyses, and quantum kinetic formulations [4, 7, 12]. The present work provides an independent and transparent semiclassical confirmation of this behavior.

Beyond reproducing known limiting results, the main contribution of the present work is pedagogical and analytic. By reducing the tunneling exponent to a real integral and expressing it in terms of complete elliptic functions, we have shown explicitly how rotational symmetry and time dependence enter the semiclassical description, and how polarization-dependent terms cancel in the adiabatic limit. This level of analytic control complements numerical and worldline based approaches, which often emphasize physical interpretation or computational efficiency at the expense of transparent intermediate steps.

Although the present treatment is restricted to an idealized homogeneous background, it provides a clean reference framework against which more complex situations, such as finite-duration pulses, spatially inhomogeneous fields, or fermionic pair production, may be compared. In this sense, the work serves both as a pedagogical guide to semiclassical methods in rotating backgrounds and as a benchmark for future extensions of strong-field QED.

We anticipate that the analytic formulation presented here will be useful for interpreting numerical studies, for testing approximations in more elaborate field configurations, and for clarifying the role of polarization in nonperturbative vacuum phenomena.

References

- [1] J. Schwinger, "On gauge invariance and vacuum polarization," *Phys. Rev.*, vol. 82, pp. 664–679, 1951.
- [2] E. Brezin and C. Itzykson, "Pair production in vacuum by an alternating field," *Phys. Rev. D*, vol. 2, no. 6, pp. 1191–1199, 1970.
- [3] V. S. Popov, "Pair production in a variable external field (quasiclassical approximation)," *Sov. Phys. JETP*, vol. 34, p. 709, 1972.
- [4] M. S. Marinov and V. S. Popov, "Electron-positron pair creation from a vacuum induced by a variable electric field," *Fortschritte der Physik*, vol. 25, no. 7, pp. 373–400, 1977.
- [5] G. V. Dunne, Q.-H. Wang, H. Gies, and C. Schubert, "Worldline instantons and the fluctuation prefactor," *Phys. Rev. D*, vol. 73, no. 6, p. 065028, 2006.
- [6] S. Strobel, S.-S. Xue, A. Blinne, F. Hebenstreit, and H. Gies, "Pair production in rotating electric fields," *Phys. Rev. D*, vol. 89, no. 6, p. 065028, 2014.
- [7] E. Strobel and S.-S. Xue, "Semiclassical pair production rate for rotating electric fields," *Phys. Rev. D*, vol. 91, p. 045016, 2015.
- [8] A. Blinne and H. Gies, "Schwinger pair production in rotating Sauter pulses," *Phys. Rev. D*, vol. 89, p. 085001, 2014.
- [9] A. Blinne, "Electron-positron pair production in strong electric fields," PhD thesis, Universität Heidelberg, 2017. arXiv:1701.00743.
- [10] O. Olugh, Z.-L. Li, B.-S. Xie, and R. Alkofer, "Pair production in differently polarized electric fields with frequency chirps," *Phys. Rev. D*, vol. 99, p. 036003, 2019.
- [11] Y. Bai and X. Song, "Electron-positron pair production in a rotating electric field," *Phys. Rev. A*, vol. 86, no. 3, p. 032110, 2012.
- [12] A. Ilderton, "Complex worldlines and radiation reaction," *Phys. Rev. D*, vol. 91, no. 10, p. 105009, 2015.
- [13] A. I. Nikishov, "On vacuum-vacuum amplitude and Bogoliubov coefficients," *J. Exp. Theor. Phys.*, vol. 95, no. 2, pp. 360–369, 2002.
- [14] G. Gamow, "Zur Quantentheorie des Atomkernes," *Z. Phys.*, vol. 51, pp. 204–212, 1928.
- [15] A. Sommerfeld, *Partial Differential Equations in Physics. Lectures on Theoretical Physics*, Vol. 6. New York: Academic Press, 1949. Reprinted by Dover Publications, 2004.
- [16] L. V. Keldysh, "Ionization in the field of a strong electromagnetic wave," *Sov. Phys. JETP*, vol. 20, no. 5, pp. 1307–1314, 1965.