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Finite element algorithm for solving one-dimensional Richards' equation

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Abstract

Accurate, efficient, robust, and stable numerical solution of Richards' equation for variably saturated-unsaturated water flow through porous media is an extremely challenging task for any numerical integrator due to its highly nonlinear behavior. In this study, we have solved a one-dimensional Richards' equation that frequently describes saturated-unsaturated water flow in homogeneous soil layers. The obtained solution is a set of an ordinary differential equation derived from mass conservation principles. We have used the finite element method for domain discretization of the governing equation, while backward Euler finite difference technique is employed for temporal discretizing to simulate infiltration and sharp fronts. As a result, mass balance errors have been reduced. The validity of the method is demonstrated with three test cases that show that the presented solution technique is explicitly integrable, has no numerical complexity. It is accurate, computationally efficient, and robust, as well as, can extend to simulate heterogeneous soils.

Keywords: Finite element; Richards' equation; Numerical solution; Saturated flow; Unsaturated flow

1. Introduction

Modeling of groundwater flow in variably saturated-unsaturated soils is of enormous concentration to various scientific explore and engineering purposes involved in agricultural engineering, groundwater management, petroleum reservoir, bioenvironmental processes, chemical contaminants tracing, etc. Richards' equation is the standard and common mathematical approach used to express the physical phenomena of water flow in a variably saturated-unsaturated porous medium, which also can be defined by the coupling of flow continuity equation with Darcy's law. A highly nonlinear characteristic of Richards' equation yields the numerical challenge, obtaining the solution in variably saturated-unsaturated water flow. Specifically, it is true, in the case of saturated soils, the behavior of the main partial differential equation transforms to parabolic from elliptical nature.

To solve Richards' equation numerically, it is required to solve the proper form of the equation, the closed formulation of the characteristic relations, the spatial, and temporal discretization techniques, linearization techniques, and the methods of solution of linear equation. Standard approaches have evolved to handle these facts, while the latest developments are recommended as attractive potential options to the typical selections in some cases.

Recently, an analytical solution of the Richards equation under gravity-driven infiltration and constant rainfall intensity is proposed [1]. Earlier, an exact solution of the Richards' equation is found based on the nonlinear plant-root extraction [2]. But very few numbers of studies have been developed for solving Richards' equation analytically. As a result, the numerical solution of Richards' equation has paid the most attention in the simulation community. In the numerical solution method, there are two major challenges will arise. The first difficulty, pressure head, and water content are two dependent variables and the constitutive relations between them are highly nonlinear. The water content-based form of Richards' equation is not applicable for the saturated flow case. Numerical solutions and appropriate form of Richards' equation can be utilized into four categories, such as the form of pressure head, Taylor series expansion of time with water content formulation, switching technique, and predictor-corrector algorithm according to their procedures conduct for the above two complexities. Pressure head-based relation, where pressure head is a dependent variable, while water content is considering the dependent variable for water content formulation and mixed form where both variables are employed, are three standard forms of Richards' equation.

Vertical one-dimensional mixed formulation of Richards' equation [3] is given as:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial \psi}{\partial z} + 1\right)\right] \qquad (1)$$

where ψ is the pressure head [L], t is time [T], z [L] signifies the positive upward vertical distance from reference elevation, $K(\psi)$ $[LT^{-1}]$, and θ are the hydraulic conductivity and moisture content respectively.