SUST J Sci Tech, Vol 30(1) 2020, 1-4 ISSN 1682-895X

Epimorphic Image of P-Ideals of P-Algebras

(Submitted: 20.01.2020 ; Accepted: 07.06.2020)

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Abstract

In this paper, we study p-ideals of a p-algebra. We prove that epimorphic image of a p-ideal is a p-ideal. Our main result is that the lattice of p-ideals of a p-algebra **L** is isomorphic to the lattice of ideals of the Boolean algebra formed by the closed elements of **L**.

Keywords: Lattices, 0-distributive lattice, pseudocomplemented lattice, ideal, filter, homomorphisms.

1. Introduction

An algebra $\mathbf{L} = \langle L; \wedge, \vee, ^*, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is called a *p-algebra* if

(i) $\mathbf{L} = \langle L; \wedge, \vee, 0, 1\rangle$ is a bounded lattice, and
(ii) for all $a \in L$, $x \leq a^*$ if and only if

$$x \wedge a = 0.$$

The bounded lattice $\mathbf{L} = \langle L; \wedge, \vee, 0, 1\rangle$ is called the *underlying lattice* of **L** and the element a^* is called the *pseudocomplement* of a. We refer the reader to [4, 5, 6, 7] for p-algebras.

The following well known identities (see [1, 2, 3, 4, 7]) are used throughout this paper.

(1) $a \leq b$ implies $b^* \leq a^*$.

(2) $a \leq a^{**}$.

(3) $a^* = a^{***}$.

(4) $(a \vee b)^* = a^* \wedge b^*$.

(5) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

(6) $a \wedge (a \wedge b)^* = a \wedge b^*$.

Let **L** and **M** be two lattices. A mapping $f: L \to M$ is called a *lattice homomorphism* if for all $x, y \in L$,

$$f(x \wedge y) = f(x) \wedge f(y)$$

and

$$f(x \vee y) = f(x) \vee f(y).$$

Let **L** and **M** be two p-algebras. A lattice homomorphism $f: \mathbf{L} \to \mathbf{M}$ is called a *homomor phism* if for all $x \in L$,

$$f(x^*) = f(x)^*,\ f(0) = 0 \text{ and } f(1) = 1.$$

A onto (lattice) homomorphism $f: \mathbf{L} \to \mathbf{M}$ is called an (*lattice*) *epimorphism*. For , *-homomorphism of semilattices we refer the reader to Blyth [1].

In section 2, we discuss epimorphic image of an ideal of a lattice. We show that lattice epimorphic image of an ideal is an ideal. We also show that lattice epimorphism preserves the operation $\wedge$ and $\vee$ in the lattice of ideals.

In Section 3, we discuss epimorphic image of p- ideals of a p-algebra. We show that epimorphic image of a p-ideal is a p-ideal. We also show that epimorphism preserves the operation $\wedge$ and $\veebar$ in the lattice of p-ideals.

In Section 4, we define an induced epimorphism. We show that if $f: \mathbf{L} \to \mathbf{M}$ is an epimorphism, then there is an epimorphism from $I^*(\mathbf{L})$ to $I^*(\mathbf{M})$ if and only if ker$f$ is a principal ideal, and the p-algebra $I_f^*(L)$ of p-ideals containing ker$f$ is isomorphic to $I_f^*(M)$. We also show that for any p-algebra **L**, $I^*(\mathbf{L}) \cong I(S(\mathbf{L}))$.

2. Epimorphic Image of an Ideal

Let **L** be a p-algebra. A non-empty subset I of L is called an *ideal* of **L** if

(i) $a \in I$ and $b \in L$ with $b \leq a$ implies $b \in I$
(ii) $a, b \in I$ implies $a \vee b \in I$.

The set of all ideals of **L** is denoted by $I(L)$. Not every homomorphic image of an ideal is an ideal.

Now we have the following result.

Lemma 2.1. Lattice epimorphic image of an ideal is an ideal.

Proof. Let $f: \mathbf{L} \to \mathbf{M}$ be an epimorphism and let I be an ideal of **L.** Let $x \in f(I)$ and $y \in M$ such that $y \leq x$. Then there is $i \in I$ and $t \in L$ such that $x = f(i)$ and $y = f(t)$. This implies

$$y = y \wedge x = f(t) \wedge f(i) = f(t \wedge i) \in f(I).$$

Now let $f(x), f(y) \in f(I)$. Then

$$f(x) \vee (y) = f(x \vee y) \in f(I).$$

Hence $f(I)$ is an ideal of **M**. □

Now we have the following result.

Lemma 2.2. Let $f: \mathbf{L} \to \mathbf{M}$ be a lattice epimorphism. Then for any $I, J \in I(L)$,

(a) $f(I \cap J) = f(I) \cap f(J)$;

(b) $f(I \vee J) = f(I) \vee f(J)$.

Proof. (a) Let $x \in f(I) \cap f(J)$. Then $x = f(i) = f(j)$ for some $i \in I$ and $j \in J$. Thus $x = f(i) \wedge f(j) = f(i \wedge j)$. So $x \in f(I \cap J)$. Hence $f(I) \cap f(J) \subseteq f(I \cap J)$. The reverse inclusion is obvious.

(b) Let $x \in f(I \vee J)$. Then $x = f(y)$ where $y \in I \vee J$. This implies $y \leq i \vee j$ for some $i \in I$ and $j \in J$. Hence

$$x = f(y) \leq f(i \vee j) = f(i) \vee f(j)$$

for some $i \in I$ and $j \in J$. This implies $x \in f(I) \vee f(J)$. Thus $f(I \vee J) \subseteq f(I) \vee f(J)$. The reverse inclusion is trivial. □

3. Epimorphic Image of a P-Ideal

An ideal I of **L** is called a *p-ideal* of **L** if

$$x \in I \Rightarrow x^{**} \in I.$$

The set of all p-ideals of a lattice **L** is denoted by $I^*(L)$. Not every homomorphic image of a p-ideal is an ideal.

Now we have the following result.

Theorem 3.1. Every epimorphic image of a p-ideal is a p-ideal.

Proof. Let $f: \mathbf{L} \to \mathbf{M}$ be an epimorphism and $I \in I^*(L)$. Suppose $f(x) \in f(I)$ and $f(t) \leq f(x)$. Then $f(t) = f(t) \wedge f(x) = f(t \wedge x) \in f(I)$. Next let $f(x), f(y) \in f(I)$. Then $f(x) \vee f(y) = f(x \vee y) \in f(I)$. Thus $f(I)$ is an ideal. Again let $f(x) \in f(I)$. Then $f(x)^{**} = f(x^{**}) \in f(I)$ as I is a p-ideal, $x^{**} \in I$. Hence $f(I)$ is a p-ideal of **M**. □

The set $I^*(L)$ forms a p-algebra which is due to [6]. The p-algebra is denoted by $I^*(\mathbf{L})$**.**

The following result is due to [6].

Theorem 3.2. $I^*(\mathbf{L}) = \langle I^*(\mathrm{L}); \wedge, \veebar, {}^*, (0], \mathrm{L} \rangle$ is a p-algebra, where

(a) $I \wedge J = I \cap J$;

(b) $I \veebar J = \{x \in L \mid x \leq (i^* \wedge j^*)^*$ for some $i \in I, j \in J\}$;

(c) $I^* = \{x \in L \mid x^{**} \wedge i = 0$ for all $i \in I\}$.

Moreover, $I^*(\mathbf{L})$ is complemented if and only if each $I \in I^*(\mathrm{L})$ is a principal ideal. □

Now we show that epimorphism preserves the $\wedge$ and the $\veebar$ of p-ideals.

Lemma 3.3. If $f: \mathbf{L} \to \mathbf{M}$ is an epimorphism, then for any $I, J \in I^*(L)$

(a) $f(I \wedge J) = f(I) \wedge f(J)$
(b) $f(I \veebar J) = f(I) \veebar f(J)$.

Proof. (a) Obvious.

(b) Let $x \in f(I \veebar J)$. Then $x = f(y)$ for some $y \in I \veebar J$. This implies $y \leq (i^* \wedge j^*)^*$ for some $i \in I$ and $j \in J$. Hence

$$x \leq f((i^* \wedge j^*)^*) = \left(f(i^*) \wedge f(j^*)\right)^* = (f(i)^* \wedge f(j)^*)^*$$

for some $i \in I$ and $j \in J$. Thus $x \in f(I) \veebar f(J)$ and hence $f(I \veebar J) \subseteq f(I) \veebar f(J)$. The reverse inclusion is trivial. □

Let $f: \mathbf{L} \to \mathbf{M}$ be an epimorphism. For any $A \subseteq M$, define

$f^{-1}(A) = \{x \in L \mid f(x) \in A\}$.

Theorem 3.4. Let $f: \mathbf{L} \to \mathbf{M}$ be an epimorphism. Then

(a) for any $J \in I(M)$, we have $f^{-1}(J) \in I(L)$;

(b) for any $J \in I^*(M)$, we have $f^{-1}(J) \in I^*(L)$.

Proof. (a) Let $J \in I(M)$. Since $f(0) = 0 \in J$, we have $f^{-1}(J)$ is non-empty. Let $x \in f^{-1}(J)$ and $t \in L$ with $t \leq x$. Then

$$f(t) = f(t \wedge x) = f(t) \wedge f(x) \leq f(x) \in J.$$

This implies $t \in f^{-1}(J)$. Now let $x, y \in f^{-1}(J)$. Then $f(x), f(y) \in J$. This implies $f(x \vee y) = f(x) \vee f(y) \in J$. Thus, $x \vee y \in f^{-1}(J)$. Hence $f^{-1}(J)$ is an ideal.

(b) Let $J \in I^*(M)$. Then by (a) $f^{-1}(J)$ is an ideal. Now let $\in f^{-1}(J)$. Then $f(x) \in J$. Since J is p-ideal $\left(f(x)\right)^{**} \in J$. This implies $f(x^{**}) \in J$. Hence $x^{**} \in f^{-1}(J)$. Thus $f^{-1}(J) \in I^*(L)$. □

4. Induced Epimorphisms

Let $f: \mathbf{L} \to \mathbf{M}$ be an epimorphism. Define $F: I^*(L) \to I^*(M)$ by $F(I) = f(I)$ and $F^{-1}: \mathrm{I}^*(M) \to \mathrm{I}^*(L)$ by $F^{-1}(J) = f^{-1}(J)$.

The map F is called an *induced map* of f and the map F^{-1} is called a *reverse induced map* of f.

Theorem 4.1. If $f: \mathbf{L} \to \mathbf{M}$ be an epimorphism, then the induced map $F: I^*(L) \to I^*(M)$ is a lattice epimorphism.

Proof. Let $I, J \in I^*(L)$. Then by Lemma 3.3,

$$F(I \cap J) = F(I) \cap F(J)$$

and

$$F(I \veebar J) = F(I) \veebar F(J).$$

Now let $J \in I^*(M)$. Since f is epimorphism, by the Theorem 3.4, $F^{-1}(J) \in I^*(L)$ and $F(F^{-1}(J)) = f(f^{-1}(J)) = J$. □

Let $f: \mathbf{L} \to \mathbf{M}$ be a homomorphism. The *kernel* of f is denoted by $\ker f$ and defined by

$$\ker f = \{x \in L \mid f(x) = 0\}.$$

Lemma 4.2. If $f: \mathbf{L} \to \mathbf{M}$ is a homomorphism, then $\ker f$ is a p-ideal.

Proof. Let $x, y \in \ker f$. Then $f(x), f(y) = 0$ and hence

$$f(x \vee y) = f(x) \vee f(y) = 0 \vee 0 = 0.$$

This implies $x \vee y \in \ker f$. Now let $x \in \ker f$ and $t \in L$ with $t \leq x$. Then $f(t) \leq f(x) = 0$ and hence $f(t) = 0$. So $\ker f$ is an ideal. Next let $x \in \ker f$. Then $f(x) = 0$. Now

$f(x^{**}) = \left(f(x)\right)^{**} = (0)^{**} = 0$. So $x^{**} \in \ker f$. Thus, $\ker f$ is a p-ideal. □

The set of all p-ideals of a p-algebra $\mathbf{L}$ that contains $\ker f$ is denoted by $I_f^*(L)$. If $\ker f = \{0\}$, then clearly, $I_f^*(L) = I^*(L)$.

Theorem 4.3. If $f: \mathbf{L} \to \mathbf{M}$ is an epimorphism, then $I_f^*(L) \cong I^*(M)$.

Proof. Clearly the map $F: I_f^*(L) \to I^*(M)$ defined above is a one-one lattice homomorphism. Let $J \in I^*(M)$. Then by Theorem 4.1, there exists $A \in I^*(L)$ such that $J = F(A)$. Now let $x \in \ker f$. Then $f(x) = 0 \in J$. So $x \in A$, that is, $\ker f \subseteq A$. Hence F is onto. □

Now we have the following important result.

Theorem 4.4. Let $f: \mathbf{L} \to \mathbf{M}$ be an epimorphism. Then for any $I \in I^*(L)$, we have

$$I \veebar \ker f = F^{-1}F(I).$$

Proof. Let $x \in I \veebar \ker f$. Then $x \leq (i^* \wedge j^*)^*$ for some $i \in I$ and $j \in \ker f$. This implies

$$\begin{aligned} f(x) &\leq f(i^* \wedge j^*)^* \\ &= \left(f(i^* \wedge j^*)\right)^* \\ &= (f(i \vee j)^*)^* \\ &= (f(i \vee j))^{**} \\ &= \left(f(i) \vee f(j)\right)^{**} \\ &= \left(f(i)\right)^{**}, \text{ since } f(j) = 0 \\ &= f(i^{**}) \in f(I) = F(I), \text{ as } I \text{ is a p-ideal.} \end{aligned}$$

Hence $x \in F^{-1}F(I)$.

Conversely, let $x \in F^{-1}F(I)$. Then $f(x) = f(i)$ for some $i \in I$. This implies $f(x) \leq f(i^{**}) = \left(f(i^*)\right)^*$ and hence $f(x \wedge i^*) = f(x) \wedge f(i^*) = 0$. Thus $x \wedge i^* \in \ker f$. Now $x \leq x^{**} \leq (i^* \wedge x^*)^* = (i^* \wedge (i^* \wedge x)^*)^*$ implies $x \in I \veebar \ker f$.

Hence $I \veebar \ker f = F^{-1}F(I)$. □

If $f: \mathbf{L} \to \mathbf{M}$ is an epimorphism, then we have $F: I^*(\mathbf{L}) \to I^*(\mathbf{M})$ is a lattice epimorphism. Next result gives us an equivalence condition of F to be an epimorphism:

Theorem 4.5. If $f: \mathbf{L} \to \mathbf{M}$ is an epimorphism then the following are equivalent:

(a) $F: I^*(\mathbf{L}) \to I^*(\mathbf{M})$ is an epimorphism;

(b) $\ker f$ is a principal ideal.

Proof. (a) ⇒ (b). Let $I = \ker f$. Then by Theorem 4.4,

$$F(I^* \veebar I) = \left(F\left(F^{-1}\left(F(I^*)\right)\right)\right) = F(I^*) = f(I^*)$$
$$= f(I)^* = (0]^* = M.$$

Since F is an epimorphism, $I \veebar I^* = L$. Hence

I^* is the complement of I in $I^*(L)$. By Theorem 3.2, I is principal.

(b) ⇒ (a). By Theorem 4.1 we have F is a lattice epimorphism. To show that F is an epimorphism we only need to show that for every $I \in I^*(L)$, $F(I^*) = F(I)^*$. Let $x \in F(I^*)$. Then $x = f(y)$ for some $y \in I^*$. So $y^{**} \wedge i = 0$ for all $i \in I$. This implies for all $i \in I$ we have

$$f(y^{**} \wedge i) = f(y^{**}) \wedge f(i) = f(y)^{**} \wedge f(i) = 0.$$

Thus $f(y) \in f(I)^* = F(I)^*$. So $F(I^*) \subseteq F(I)^*$.

Conversely, let $x \in F(I)^*$. Then $x^{**} \wedge f(i) = 0$ for all $i \in I$ This implies $x \wedge f(i) \leq x^{**} \wedge f(i) = 0$. Now $x \in M$ implies that $x = f(z)$ for some $z \in L$. So we get $f(z) \wedge f(i) = f(z \wedge i) = 0$. Thus $z \wedge i \in \ker f$. Since $\ker f$ is a principal ideal we have $\ker f = (t^{**}]$ for some $t \in L$.

Now for all $i \in I$,

$$\begin{aligned} & z \wedge i \leq t^{**} \\ \Rightarrow\ & z \wedge i \wedge t^* = 0 \\ \Rightarrow\ & (z \wedge i \wedge t^*)^{**} = 0 \\ \Rightarrow\ & (z \wedge t^*)^{**} \wedge i^{**} = 0 \\ \Rightarrow\ & (z \wedge t^*)^{**} \wedge i = 0 \\ \Rightarrow\ & z \wedge t^* \in I^*. \end{aligned}$$

Since $t \in \ker f$, we have

$$f(z \wedge t^*) = f(z) \wedge f(t^*) = f(z) \wedge \left(f(t)\right)^*$$
$$= f(z) \wedge 1 = f(z) = x.$$

So $x \in F(I^*)$. Thus $F(I^*) = F(I)^*$. This completes the proof. □

The following result is due to [6].

Theorem 4.6. Let **L** be a p-algebra. Then the following conditions are equivalent:

(a) Every ideal is a p-ideal;

(b) Every principal ideal is a p-ideal;

(c) **L** is a Boolean algebra. □

It is well known that the set $S(L) = \{x^{**} \mid x \in L\}$ of closed elements of a p-algebra **L** form a Boolean algebra $S(\mathbf{L}) = \langle S(L); \wedge, \sqcup, {}^*, 0, 1\rangle$ where for any $a, b \in S(L)$, $a \sqcup b = (a^* \wedge b^*)^*$.

Finally we close the paper with the following result.

Theorem 4.7. For any p-algebra **L**, we have

$$I^*(\mathbf{L}) \cong I(S(\mathbf{L})).$$

Proof. Since $S(L)$ is Boolean, by Theorem 4.6 we have $I^*(S(L)) = I(S(L))$. Let $g: \mathbf{L} \to S(\mathbf{L})$ be the Glivenko epimorphism defined by $g(x) = x^{**}$. Then by Theorem 4.3, $I_f^*(\mathbf{L}) \cong I^*(S(\mathbf{L}))$. Now since $\ker g = \{0\}$, we have $I_f^*(\mathbf{L}) = I^*(\mathbf{L})$. Hence $I^*(\mathbf{L}) \cong I(S(\mathbf{L}))$. □

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