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Annihilator Ideals in 0-distributive Lattices

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Abstract

In this paper we generalized some results of annihilator ideals for 0-distributive lattices. We prove that the set of all annihilator ideals of a 0-distributive lattice form a Boolean algebra. We also prove Stone type separation theorem for annihilator ideals of 0-distributive lattice.

Keywords: 0-distributive lattice, annihilator, ideal, Boolean algebra.

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1. Introduction

A lattice L with 0 is called 0-distributive if for any a, b, c $\in$ L such that $a \wedge b = 0 = a \wedge c$ implies $\wedge (b \vee c) = 0$. Varlet [7] introduced the notion of 0-distributive lattice for generalization of distributive lattices. For the background of lattice theory we refer the reader to the foundation monograph [3].

Annihilator ideals play an important role to study lattice theory. Cornish [2] and Davey [1] studied annihilator ideals for distributive lattices. In this paper we study annihilator ideals of a 0-distributive lattice. We generalize some results of distributive lattices.

In Section 2, we prove some identities for annihilators which we need in this paper.

Pawar and Khopade [5] have studied α-ideals and annihilator ideals of a 0-distributive lattice as a consequence study of [6, 4]. They have mentioned that the set of all annihilator ideals of a 0-distributive lattice form a Boolean algebra. The lattice of all annihilator ideals is a sublattice of the lattice of all α-ideals of a 0-distributive lattice. In Section 3, we discuss annihilator ideals of a 0-distributive lattice. We prove that the set of all annihilator ideals of a 0-distributive lattice form a Boolean algebra. Our definition of supremum of two annihilator ideals is different from the definition of supremum of two α-ideals given in [5].

In Section 4, we found that prime ideals and annihilator ideals of a 0-distributive lattice are independent. In this section we show that a non-dense prime ideal of a 0-distributive lattice is an annihilator ideal. We also give a characterization of prime ideal in 0-distributive lattices to be an annihilator ideal. We establish a sufficient condition for an annihilator ideal to be a prime ideal. We prove Stone type separation theorem for annihilator ideals of 0-distributive lattices.

2. Annihilators

Let $\boldsymbol{L}$ be a 0-*distributive* lattice and A be a non-empty subset of L. Define

$A^{\perp} = \{x \in S \mid x \wedge a = 0 \text{ for all } a \in A\}$.

Then $A^{\perp}$ is called the *annihilator* of A. If $a \in A$ then the annihilator of {a} is denoted by $a^{\perp}$ and defined as

$a^{\perp} = \{x \in L \mid x \wedge a = 0\}$.

It is called the *annulet* generated by a.

The following identities are easy to prove and will be used throughout this paper.

Lemma 2.1. Let $\boldsymbol{L}$ be a lattice with 0 and $a, b \in L$. Then

(i) $a \in a^{\perp\perp}$.

(ii) If $a \leq b$, then $b^{\perp} \subseteq a^{\perp}$ and $a^{\perp\perp} \subseteq b^{\perp\perp}$

(iii) $a^{\perp\perp\perp} = a^{\perp}$.

(iv) $a^{\perp} \cap a^{\perp\perp} = \{0\}$.

Now we have the following useful result.

Lemma2.2. Let $\boldsymbol{L}$ be a 0-distributive lattice and $a, b \in L$. Then $(a \wedge b)^{\perp\perp} = a^{\perp\perp} \cap b^{\perp\perp}$.

Proof. Let $x \in a^{\perp\perp} \cap b^{\perp\perp}$ and $y \in (a \wedge b)^{\perp}$. Then $y \wedge a \wedge b = 0$. This implies $y \wedge a \in b^{\perp}$. Since $x \in b^{\perp\perp}$, we have $x \wedge y \wedge a = 0$. This implies $x \wedge y \in a^{\perp}$. Again since $x \in a^{\perp\perp}$, so $x \wedge y \in a^{\perp\perp}$. Thus $x \wedge y \in a^{\perp} \cap a^{\perp\perp} = 0$. This implies $x \wedge z = 0$ for all $z \in (a \wedge b)^{\perp}$. Then $x \in (a \wedge b)^{\perp\perp}$ and hence $a^{\perp\perp} \cap b^{\perp\perp} \subseteq (a \wedge b)^{\perp\perp}$. Converse is due to Lemma 2.1 (ii). Hence $(a \wedge b)^{\perp\perp} = a^{\perp\perp} \cap b^{\perp\perp}$.

3. Annihilator Ideals

Let L be a 0-distributive lattice. A non-empty subset F of L is called a *filter* if

(i) $a \in F$ and $b \in L$ with $b \geq a$ implies $b \in F$,

(ii) $a, b \in F$ implies $a \wedge b \in F$.

For any $a \in L$, the set

$$[a) = \{x \in L \mid a \leq x\}$$

is a filter called the *principal filter* generated by a.

A non-empty subset I of L is called an *ideal* if

(i) $a \in I$ and $b \in L$ with $b \leq a$ implies $b \in I$

(ii) $a, b \in I$ implies $a \vee b \in I$.

For any $a \in L$, the set

$$(a] = \{x \in L \mid x \leq a\}$$

is an ideal which is called the *principal ideal* generated by a.

The set of all ideals of L is denoted by $I(L)$. It is well known that $\langle I(L), \subseteq\rangle$ forms a lattice as an ordered set. For $I, J \in I(L)$ we denote

$$I \wedge J = \inf_{I(L)}\{I, J\} \text{ and } I \vee J = \sup_{I(L)}\{I, J\}.$$

Now we have the following useful properties for $I(L)$.

Lemma 3.1. Let L be a lattice with 0 and

$I, J \in I(L)$. Then

(i) $I \wedge J = (0]$ if and only if $J \subseteq I^{\perp}$.

(ii) $I \cap I^{\perp} = (0]$.

(iii) $I \subseteq I^{\perp\perp}$.

(iv) If $I \subseteq J$, then $J^{\perp} \subseteq I^{\perp}$.

(v) $I^{\perp\perp\perp} = I^{\perp}$.

Proof. (i) Let $b \in J$. Then $a \wedge b = 0$ for all $a \in I$, as $I \wedge J = (0]$. Thus $b \in I^{\perp}$. Hence $J \subseteq I^{\perp}$. Converse is obvious.

(ii) Putting $J = I^{\perp}$ in (i).

(iii) Using (i) and (ii).

(iv) Let $x \in J^{\perp}$. Then $x \wedge j = 0$ for all $j \in J$. Since $I \subseteq J$, therefore $x \wedge i = 0$ for all $i \in I$. Hence $J^{\perp} \subseteq I^{\perp}$.

(v) By (iii), $I^{\perp} \subseteq I^{\perp\perp\perp}$. Converse is true by (iii) and (iv). Hence $I^{\perp\perp\perp} = I^{\perp}$.

Now we have the following useful result for ideals.

Lemma 3.2. Let L be a 0-distributive lattice and $I, J \in I(L)$. Then $(I \wedge J)^{\perp\perp} = I^{\perp\perp} \wedge J^{\perp\perp}$.

Proof. Let $x \in I^{\perp\perp} \wedge J^{\perp\perp}$ and $y \in (I \wedge J)^{\perp}$. Then $(y \wedge i) \wedge j = 0$ for all $i \in I$ and $j \in J$. This implies $y \wedge i \in j^{\perp}$ for all $j \in J$. Hence $y \wedge i \in J^{\perp}$. Since $x \in J^{\perp\perp}$, we get $(x \wedge y) \wedge i = 0$ for all $i \in I$. Hence $x \wedge y \in I^{\perp}$. Since L is 0-distributive, we have $I^{\perp\perp} \in I(L)$. Now $x \in I^{\perp\perp}$ implies $x \wedge y \in I^{\perp\perp}$. Thus $x \wedge y \in I^{\perp} \wedge I^{\perp\perp} = (0]$. Hence $x \wedge y = 0$ for all $y \in (I \wedge J)^{\perp}$. Therefore $x \in (I \wedge J)^{\perp\perp}$. Hence $I^{\perp\perp} \wedge J^{\perp\perp} \subseteq (I \wedge J)^{\perp\perp}$. The converse follows from Lemma 3.1 (iv). Hence $(I \wedge J)^{\perp\perp} = I^{\perp\perp} \wedge J^{\perp\perp}$.

Lemma 3.3. Let L be a 0-distributive lattice and $I, J \in I(L)$. Then $(I \vee J)^{\perp} = I^{\perp} \wedge J^{\perp}$.

Proof. Since $I, J \subseteq I \vee J$, so we have $(I \vee J)^{\perp} \subseteq I^{\perp} \wedge J^{\perp}$. On the other hand let $x \in I^{\perp} \wedge J^{\perp}$. Then $x \wedge t = 0$ for all $t \in I$ and $x \wedge s = 0$ for all $s \in J$. By the property of 0-distributivity of L we have $x \wedge (t \vee s) = 0$. This implies $x \wedge i = 0$ for all $i \in I \vee J$. Hence $x \in (I \vee J)^{\perp}$. Thus $(I \vee J)^{\perp} = I^{\perp} \wedge J^{\perp}$.

Let L be a 0-distributive lattice. Then $A^{\perp}$ is an ideal of L for any non-empty subset A of L. An ideal I of L is called an *annihilator ideal* if $I = A^{\perp}$ for some non-empty subset A of L or equivalently if $I = I^{\perp\perp}$. The set of all annihilator ideals of L is denoted by $\mathcal{A}(L)$. If $I, J \in \mathcal{A}(L)$, then by Lemma 3.2, $I \wedge J \in \mathcal{A}(L)$. Observe that $I \vee J$ may not be an annihilator ideal. For counterexample, consider the lattice $\mathcal{M}$ in Figure 1. Then $I = (a]$ and $J = (b]$ are annihilator ideals but $I \vee J = (c]$ is not an annihilator ideal. Thus $\langle \mathcal{A}(L), \subseteq \rangle$ is not a sublattice of $\langle I(L), \subseteq\rangle$. Now we show that $\langle \mathcal{A}(L), \subseteq\rangle$ is itself a lattice as an ordered set. For any $I, J \in \mathcal{A}(L)$ we denote $I \sqcap J = inf_{\mathcal{A}(L)}\{I, J\}$ and $I \sqcup J = sup_{\mathcal{A}(L)}\{I, J\}$. Indeed, we have the following result.

Theorem 3.4. Let $\boldsymbol{L}$ be a 0-distributive lattice. For any $I, J \in \mathcal{A}(L)$ we have

(a) $I \sqcap J = I \wedge J$;

(b) $I \sqcup J = (I^{\perp} \wedge J^{\perp})^{\perp}$.

Proof. (a) Obvious.

(b) Clearly $I^{\perp} \wedge J^{\perp} \subseteq I^{\perp}, J^{\perp}$. This implies $I^{\perp\perp} = I \subseteq (I^{\perp} \wedge J^{\perp})^{\perp}$ and $J^{\perp\perp} = J \subseteq (I^{\perp} \wedge J^{\perp})^{\perp}$. Hence $I \sqcup J \subseteq (I^{\perp} \wedge J^{\perp})^{\perp}$. Now let K be an annihilator ideal such that $I, J \subseteq K$. Then $K^{\perp} \subseteq I^{\perp} \wedge J^{\perp}$. Since K is an annihilator ideal, this implies $(I^{\perp} \wedge J^{\perp})^{\perp} \subseteq K^{\perp\perp} = K$. Hence $I \sqcup J = (I^{\perp} \wedge J^{\perp})^{\perp}$.

The above theorem shows that $\langle \mathcal{A}(L), \sqcup, \sqcap \rangle$ forms a lattice.

Lemma 3.5. Let $\boldsymbol{L}$ be a 0-distributive lattice and $I, J, K \in \mathcal{A}(L)$. Then $(I \sqcup J) \sqcap K \subseteq I \sqcup (J \sqcap K)$.

Proof. We have

$I \sqcap K \sqcap [I^{\perp} \sqcap (J \sqcap K)^{\perp}] = (0]$ and

$J \sqcap K[I^{\perp} \sqcap (J \sqcap K)^{\perp}] = (0]$

$\Rightarrow K \sqcap I^{\perp} \sqcap (J \sqcap K)^{\perp} \subseteq I^{\perp}$ and

$K \sqcap I^{\perp} \sqcap (J \sqcap K)^{\perp} \subseteq J^{\perp}$

$\Rightarrow K \sqcap I^{\perp} \sqcap (J \sqcap K)^{\perp} \subseteq I^{\perp} \sqcap J^{\perp}$

$\Rightarrow K \sqcap I^{\perp} \sqcap (J \sqcap K)^{\perp} \sqcap (I^{\perp} \sqcap J^{\perp})^{\perp} = (0]$

by Lemma 3.1(i)

$\Rightarrow I^{\perp} \sqcap (J \sqcap K)^{\perp} \sqcap [K \sqcap (I^{\perp} \sqcap J^{\perp})^{\perp}] = (0]$

$\Rightarrow K \sqcap (I^{\perp} \sqcap J^{\perp})^{\perp} \subseteq [I^{\perp} \sqcap (J \sqcap K)^{\perp}]^{\perp}$

by Lemma 3.1(i)

Hence by Theorem 3.4, we have

$$(I \sqcup J) \sqcap K \subseteq I \sqcup (J \sqcap K).$$

The following theorem shows that $\langle \mathcal{A}(L), \sqcup, \sqcap \rangle$ is a distributive lattice.

Theorem 3.6. Let $\boldsymbol{L}$ be a 0-distributive lattice. Then $\langle \mathcal{A}(L), \sqcup, \sqcap \rangle$ is a distributive lattice.

Proof. For any ideals $I, J, K \in \mathcal{A}(L)$,

$(I \sqcup J) \sqcap (I \sqcup K) \subseteq I \sqcup [J \sqcap (I \sqcup K)]$,

by Lemma 3.5

$\subseteq I \sqcup I \sqcup (J \sqcap K)$,

by Lemma 3.5

$= I \sqcup (J \sqcap K)$.

The reverse inclusion is trivial. Hence $\langle \mathcal{A}(L), \sqcup, \sqcap \rangle$ is a distributive lattice.

Clearly, $0^{\perp} = \boldsymbol{L}$ and $L^{\perp} = (0]$. Now we have the following result.

Theorem 3.7. Let $\boldsymbol{L}$ be a 0-distributive lattice. Then $\langle \mathcal{A}(L); \sqcup, \sqcap, {}^{\perp}, (0], L \rangle$ is a Boolean Algebra.

Proof. We have $\langle \mathcal{A}(L); \sqcup, \sqcap, (0], L \rangle$ is a bounded distributive lattice. Now for any $I \in \mathcal{A}(L)$, we have

$$I \sqcap I^{\perp} = (0]$$

and

$$I \sqcup I^{\perp} = (I^{\perp} \sqcap I^{\perp\perp})^{\perp} = (0]^{\perp} = \mathbb{L}.$$

Hence $\langle \mathcal{A}(L); \sqcup, \sqcap, {}^{\perp}, (0], L \rangle$ is a Boolean Algebra.

4. Separation Theorem for Annihilator Ideals

A *prime ideal* P is a proper ideal of $\boldsymbol{L}$ such that $a \wedge b \in P$ implies either $a \in P$ or $b \in P$. We denote the set of all prime ideals of $\boldsymbol{L}$ by $P_I(L)$. A prime ideal P is called *minimal*, if for any prime ideal $Q \subseteq P$ implies $P = Q$. A filter F of $\boldsymbol{L}$ is called a *prime filter* if $F \neq L$ and $L \setminus F$ is a prime ideal. It is well known that a filter F is a maximal filter if and only if $L \setminus F$ is a minimal prime ideal.

We observed in $\mathcal{M}$ (see Figure 1) that $I = (0]$ is a annihilator ideal but not prime. On the other hand $I = (c]$ is a prime ideal but not annihilator ideal. Thus the prime ideals and annihilator ideals of a 0-distributive lattice are independent.

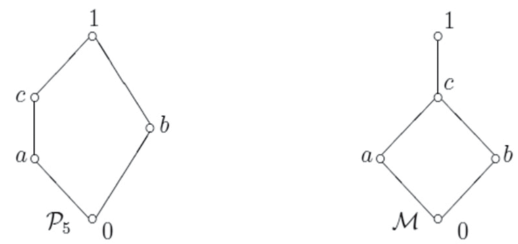

FIGURE 1. Two important 0-distributive lattices

Let $\boldsymbol{L}$ be a 0-distributive lattice. An element $d \in L$ is called *dense* if $d^{\perp} = (0]$. The set of all dense element of L is denoted by $\mathcal{D}(L)$. Thus

$$\mathcal{D}(L) = \{x \in L \mid x^{\perp} = (0]\}.$$

It is well known that $\mathcal{D}(L)$ is a filter. An ideal I of $\boldsymbol{L}$ is called a *dense ideal* if $I^{\perp} = (0]$.

Lemma 4.1. Let $\boldsymbol{L}$ be a 0-distributive lattice. Then a proper annihilator ideal contains no dense element.

Proof. Let I be a proper annihilator ideal of $\boldsymbol{L}$ and $x \in I \cap \mathcal{D}(L)$. Then $x^{\perp} = (0]$. This implies $I^{\perp} = (0]$. Therefore I is a dense ideal and $I = I^{\perp\perp} = L$, which contradicts the fact that I is proper. Therefore $I \cap \mathcal{D}(L) = \phi$.

Theorem 4.2. If a prime ideal P of a 0-distributive lattice L is non- dense, then P is an annihilator ideal.

Proof. Let P be a prime ideal of $\boldsymbol{L}$. If P is non-dense, then $P^{\perp} \neq (0]$. This implies there exists $0 \neq x \in L$ such that $x \in P^{\perp}$. Hence $P^{\perp\perp} \subseteq x^{\perp}$ and so $P \subseteq x^{\perp}$. We show that $P = x^{\perp}$. If not, then let $a \in x^{\perp} \setminus P$. This implies $a \wedge x = 0 \in P$. Thus $x \in P$ as P is a prime ideal. This shows that $x \in P \cap P^{\perp} = (0]$, which is a contradiction. Hence $x^{\perp} = P$. Therefore P is an annihilator ideal.

The above result gives us the following corollary.

Corollary 4.3. A prime ideal P of a 0-distributive lattice $\boldsymbol{L}$ is an annihilator ideal if and only if $P \cap \mathcal{D}(L) = \phi$.

Now we have the following theorem.

Theorem 4.4 (Separation Theorem for Annihilator Ideals). Let $\boldsymbol{L}$ be a 0-distributive lattice, I be an annihilator ideal and F be a filter of $\boldsymbol{L}$ such that $I \cap F = \phi$. Then there is a prime annihilator ideal containing I and disjoint from F.

Proof. Let I be an annihilator ideal and F be a filter of a lattice $\boldsymbol{L}$ such that $I \cap F = \phi$. Define

$$\chi = \{G \mid G \text{ is a filter of } \boldsymbol{L} \text{ such that } F \subseteq G \text{ and } I \cap G = \phi\}.$$

Clearly, $F \in \chi$ and χ satisfies all the conditions of Zorn's Lemma. Thus χ has a maximal element, say M. We show that M is a prime filter. Suppose M is not prime. Then there are $a, b \in L \setminus M = P$ such that $a \vee b \in M$. By the maximality of M in χ, we have $\big(M \vee [a)\big) \cap I \neq \phi$ and $\big(M \vee [b)\big) \cap I \neq \phi$. Let $x \in \big(M \vee [a)\big) \cap I$ and $y \in \big(M \vee [b)\big) \cap I$. Then $x, y \in I$ such that $x \geq m \wedge a$ and $y \geq n \wedge b$ for some $m, n \in M$. Since I is an annihilator ideal, $m \wedge a \in I = I^{\perp\perp}$ and $n \wedge b \in I = I^{\perp\perp}$. This implies $m \wedge a \wedge i = 0$ and $n \wedge b \wedge i = 0$ for all $i \in I^{\perp}$. Hence $(m \wedge n \wedge i) \wedge a = 0 = (m \wedge n \wedge i) \wedge b$. Since $\boldsymbol{L}$ is 0-distributive, we have $(m \wedge n \wedge i) \wedge (a \vee b) = 0$ for all $i \in I^{\perp}$. Thus $(m \wedge n) \wedge (a \vee b) \in I^{\perp\perp} = I$. Which is a contradiction to the fact that $I \cap M = \phi$. Therefore, M is a prime filter and hence $P = L \setminus M$ is a prime ideal.

Finally we claim that P is an annihilator ideal. It is enough to show that P contains no dense element. If $x \in P \cap \mathcal{D}(L)$. Then $x \notin M$ and by the maximality of M with I, we have $\big(M \vee [x)\big) \cap I \neq \phi$. Let us consider $y \in \big(M \vee [x)\big) \cap I$. Then $y \geq t \wedge x$ for some $t \in M$. Since $y \in I$, therefore $t \wedge x \in I \subseteq P$. Thus by Lemma 2.2 $(t \wedge x)^{\perp\perp} = t^{\perp\perp} \cap x^{\perp\perp} \subseteq I^{\perp\perp} = I$, as I is an annihilator ideal. Since x is a dense element $x^{\perp\perp} = L$, therefore $t^{\perp\perp} \subseteq I$. Since $t \in t^{\perp\perp}$, therefore $t \in I$. This contradicts the fact that $M \cap I = \phi$. Therefore $P \cap \mathcal{D}(L) = \phi$ and hence by Theorem 4.3, P is the required prime annihilator ideal.

References:

1. Davey, B.A., Some Annihilator Conditions on Distributive Lattices, Algebra Universalis, 1974, 4(1), 316 – 322.
2. Cornish, W.H., Annulets and α-ideals in a Distributive Lattices, Journal of the Australian Mathematical Society, 1973, 15(1), 70 – 77.
3. Gratzer, G., General Lattice Theory, Birkhauser Verlag Basel, 1998.
4. Jayaram, C., Prime α-ideals in 0-distributive Lattice, Indian J. Pure Appl. Math. 1986, 17(3), 331 – 337.
5. Pawar, Y. S., and Khopode, S. S., α-ideals and Annihilator Ideals in 0-distributive Lattices, Acta Univ. Palacki. Olomuc., Fac. Rer. Nat., Mathematica 2010, 49(1), 63 – 74.
6. Pawar, Y. S. and Mane, D. N., α-ideals in 0-distributive semilattices and 0-distributive Lattices, Indian J. Pure Appl. Math., 1993, 24(7-8), 435 – 443.
7. Varlet, J.C., A Generalization of the Notion of Pseudo-complementedness. Bulletin de ln Societe des Sciences de Liege, 1967, 36, 149 – 158.