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Finite element formulation employing higher order elements and software for one dimensional engineering problems

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Abstract

This paper concentrates to present finite element formulation of equilibrium problem in one dimension employing lower to higher order finite elements and the development of the relevant software based on the explicit schemes. Present formulation is new and the elements of order one to tenth are considered and element matrices are presented in such a way that the code in any computer language can be developed easily. Finally, the efficiency and accuracy tested through the demonstration of application examples.

Keywords: Finite element formulation, Shape functions, Ordinary differential equation, Boundary value problem.

1. Introduction

While searching for a quantitative description of physical phenomena, the engineer or the physicist establishes generally a system of ordinary or partial differential equations valid in a certain region and imposes on this system suitable boundary and initial conditions. At this stage the mathematical model is complete, and for practical applications "merely" a solution for a particular set of numerical data is needed. Here, however, come the major difficulties, as only the very simplest forms of equations, within geometrically trivial boundaries, are capable of being solved exactly with available mathematical tools. Ordinary differential equations with constant coefficients are one of the few examples for which standard solution procedures are available- and- even here, with a large number of dependent variables considerable difficulties are encountered [1-5]. To overcome such difficulties and to enlist the aid of the most powerful tool developed in this century-the digital computer-it is necessary to recast the problem in a purely algebraic form, involving only the basic arithmetic operations. To achieve this various forms of discretizations of the continuum problem defined by the differential equations can be used. Today the finite element method is a powerful tool for the approximate solution of differential equations governing divers physical phenomena. Its use in industry and research is extensive, and indeed it could be said that without it many problems would be incapable of solution [1-7]. In this note general finite element formulation of one dimensional boundary value problem generally known as the field problem is presented. The formulation is based on the Galerkin's approach and that utilizes lower to higher order elements. Element matrices are presented in explicit form that require some symbolic manipulation and is very easy for computation. In a word, the integration is not at all needed as it is done forever. Now, one can evaluate the expressions as presented with minimum effort. A computer code, which is compatible with the formulation, is developed and its test is demonstrated through practical applications and the best accuracy found for the higher order element formulation. Thus, we believe that the present work will lead to an efficient evaluation of element matrices and this may be used as the analysis tools by the users specially physicists and engineers.

The developed code based on the present formulation is user friendly and requires minimum number of inputs for whole computations. Different lengths of the element may be used or if user desires to use the elements of same length then one time computation of element matrices are needed and this substantially reduces the computing time. With due respect to the users choice both the options are kept in usable conditions. We wish to note here that if the domain of the problem is considered as one element then higher order element formulation of this note is recommended to use and the best accuracy may be achieved instead of using a huge number of lower order elements in descretization. This was substantiated by the solution of many physical problems. Another fact that is where the function variation occurs frequently higher order elements give rise to a satisfactory results.

2. Finite Element Formulation of One-dimensional equilibrium Problem

The governing equation together with boundary conditions are as given below:

$$D\frac{d^2u}{dx^2}+Gu+Q=0 \qquad a\le x\le b \qquad (1)$$

The boundary condition is

$$\alpha_0 u(a)+\alpha_1 u'(a)=\gamma_1,\ \beta_0 u(b)+\beta_1 u'(b)=\gamma_2$$

The trial solution is $\bar{u}=\theta_0(x)+\sum_{i=1}^{n}\alpha_i\theta_i(x)$

Corresponding weighted residual equation is given by

$$\int_a^b\left(D\frac{d^2u}{dx^2}+Gu+Q=0\right)\theta_i(x)=0 \qquad i=1,2,.......,n$$

We obtain

$$D\left[\theta_i(b)u'(b)-\theta_i(a)u'(a)\right]-D\int_a^b\left(\frac{du}{dx}\frac{d\theta_i}{dx}\right)dx + \int_a^b\left[Gu\theta_i+Q\theta_i\right]dx=0 \qquad (A)$$

There are four different cases for the general boundary conditions and Galerkin's approach give rise the following:

(i) $\alpha_1\ne 0,\beta_1\ne 0$ (both boundary condition(b.c) suppressible)

$$u'(a)=\frac{1}{\alpha_1}\left[\gamma_1-\alpha_0 u(a)\right],$$
$$u'(b)=\frac{1}{\beta_1}\left[\gamma_2-\beta_0 u(b)\right]$$
$$B_i=N_i(b)\frac{1}{\beta_i}\{\gamma_2-\beta_0 u(b)\}-N_i(a)\frac{1}{\alpha_i}\{\gamma_1-\alpha_0 u(a)\}$$

(ii) $\alpha_1\ne 0,\beta_1=0$ (suppressible boundary condition at x = a and essential b.c at x = b)

$$u'(a)=\frac{1}{\alpha_1}\left[\gamma_1-\alpha_0 u(a)\right],$$
$$\beta_0 u(b)=\gamma_2,$$
$$\theta_i(b)=0, \qquad i=1,2,\cdots,n$$

$$B_i=N_i(a)\frac{1}{\alpha_i}\{\gamma_1-\alpha_0 u(a)\}$$

(iii) $\alpha_1=0,\beta_1\ne 0$ (suppressible boundary condition at x = b and essential b.c at x = a)

$$\alpha_0 u(a)=\gamma_1,$$
$$u'(b)=\frac{1}{\beta_1}\left[\gamma_2-\beta_0 u(b)\right],\theta_i(a)=0, \qquad i=1,2,\cdots,n$$

$$B_i=N_i(b)\frac{1}{\beta_i}\{\gamma_2-\beta_0 u(b)\}$$

(iv) $\alpha_1=0,\beta_1=0$ (both boundary condition essential)

$$\alpha_0 u(a)=\gamma_1,\ \beta_0 u(b)=\gamma_2,\ \theta_i(a)=0,$$
$$\theta_i(b)=0, \qquad i=1,2,.\cdots,n$$
$$B_i=0$$

And for all the cases the matrix form of the eq.(A) is given by

$$D[k_{ij}^D]-G[k_{ij}^G]=Q\{F_i\}+D\{B_i\}$$

where

$$k_{ij}^D=\int_a^b\frac{dN_i}{dx}\frac{dN_j}{dx}dx,$$
$$k_{ij}^D=\int_a^b N_iN_jdx,$$
$$F_i=\int_a^b N_idx$$

and $N_i(\xi)$ for $i=1,NP$ (B)

are shape functions, NP denotes nodal points specified on the element. Now our aim is to evaluate the components of above matrices in eq.(B).

3. General form of shape functions and component of element matrices

$N_i(\xi)=\sum_{k=0}^{NP-1}E_{ki}\xi^k \quad i=1,NP$, E_{ki} are existence constants defined explicitly by the well known knocker delta.

Formation of element matrices in eq.(B) requires the evaluation of the following integrals in the local space:

$$K_{ij}^G=\frac{L}{2}\int_{-1}^{1}N_iN_jd\xi,\ K_{ij}^D=\frac{2}{L}\int_{-1}^{1}\frac{dN_i}{d\xi}\frac{dN_j}{d\xi}d\xi,$$
$$F_i=\frac{L}{2}\int_{-1}^{1}N_id\xi \text{ for } i,j=1,NP$$

where L is the length of the element. Existence constants and components of element matrices evaluated as follows:

3.1 For linear element or first order element (NP=2)

$$E_{0i} = \frac{1}{2}\left(\delta_{1i} + \delta_{2i}\right), E_{1i} = \frac{1}{2}\left(\delta_{2i} - \delta_{1i}\right), K_{ij}^{G} = \frac{L}{3}\left(3E_{0i}E_{0j} + E_{1i}E_{1j}\right)$$

$$K_{ij}^{D} = \frac{4}{L}E_{1i}E_{1j}, F_i = LE_{0i} \quad \text{for } i, j = 1, NP$$

Above expressions are generally written for computing the components of element stiffness matrices, load and force vectors. One can explicitly evaluate these components using the property of knocker delta. For computer evaluation, these expressions are suitable and hence for coding these will be used. Similar expressions for other higher order elements are presented in the subsequent sections using the respective shape functions. The order of element stiffness matrices are (NP,NP) and the order of the load and force vectors are (1,NP) and that should declared as the dimension of such arrays.

3.2 For quadratic element or second order element (NP=3)

$$E_{0i} = \delta_{2I}, E_{1i} = \frac{1}{2}\left(\delta_{3i} - \delta_{1i}\right), E_{2i} = \frac{1}{2}\left(\delta_{1i} - 2\delta_{2i} + \delta_{3i}\right)$$

$$K_{ij}^{G} = \frac{L}{15}(15E_{0i}E_{0j} + 5E_{0i}E_{2j} + 5E_{2i}E_{0j} + 5E_{1i}E_{1j} + 3E_{2i}E_{2j})$$

$$K_{ij}^{D} = \frac{4}{3L}\left(3E_{1i}E_{1j} + 4E_{2i}E_{2j}\right), F_i = \frac{L}{3}\left(3E_{0i} + E_{2i}\right) \quad i, j = 1, NP$$

3.3 For cubic element or third order element (NP=4)

$$E_{0i} = \frac{1}{16}\left(-\delta_{1i} + 9\delta_{2i} + 9\delta_{3i} - \delta_{4i}\right)$$

$$E_{1i} = \frac{1}{16}\left(\delta_{1i} - 27\delta_{2i} + 27\delta_{3i} - \delta_{4i}\right)$$

$$E_{2i} = \frac{9}{16}\left(\delta_{1i} - \delta_{2i} - \delta_{3i} + \delta_{4i}\right)$$

$$E_{3i} = \frac{9}{16}\left(-\delta_{1i} + 3\delta_{2i} - 3\delta_{3i} + \delta_{4i}\right)$$

$$K_{ij}^{G} = \frac{L}{105}(105E_{0i}E_{0j} + 35E_{2i}E_{0j} + 35E_{1i}E_{1j} + 21E_{3i}E_{1j} + 35E_{0i}E_{2j} + 21E_{2i}E_{2j} + 21E_{1i}E_{3j} + 15E_{3i}E_{3j})$$

$$K_{ij}^{D} = \frac{4}{15L}(15E_{1i}E_{1i} + 15E_{3i}E_{1j} + 20E_{2i}E_{2j} + 15E_{1i}E_{3j} + 27E_{3i}E_{3j})$$

$$F_i = \frac{L}{3}\left(3E_{0i} + E_{2i}\right) \quad \text{for } i, j = 1, NP$$

3.4 For fourth order element (NP=5)

$$E_{0j} = \delta_{3i}, E_{1j} = \frac{1}{6}(\delta_{1i} - 8\delta_{2i} + 8\delta_{4i} - \delta_{5i}),$$

$$E_{2j} = \frac{1}{6}(-\delta_{1i} + 16\delta_{2i} - 30\delta_{3i} + 16\delta_{4i} - \delta_{5i}),$$

$$E_{3j} = \frac{4}{6}(-\delta_{1i} + 2\delta_{2i} - 2\delta_{4i} + \delta_{5i}),$$

$$E_{4j} = \frac{4}{6}(\delta_{1i} - 4\delta_{2i} + 6\delta_{3i} - 4\delta_{4i} + \delta_{5i})$$

$$K_{ij}^{G} = \frac{L}{315}\begin{bmatrix} 21E_{0j}(15E_{0i} + 5E_{2i} + 3E_{4i}) \\ +21E_{1j}(5E_{1i} + 3E_{3i}) + 3E_{2j} \\ (35E_{0i} + 21E_{2i} + 15E_{4i}) \\ +9E_{3j}(7E_{1i} + 5E_{3i}) + \\ E_{4j}(63E_{0i} + 45E_{2i} + 35E_{4i}) \end{bmatrix}$$

$$K_{ij}^{D} = \frac{4}{105L}\begin{bmatrix} 105E_{1j}(E_{1i} + E_{3i}) + 28E_{2j} \\ (5E_{2i} + 6E_{4i}) + 21E_{3j} \\ (5E_{1i} + 9E_{3i}) + 24E_{4j} \\ (7E_{2i} + 10E_{4i}) \end{bmatrix}$$

$$F_i = \frac{L}{15}(15E_{0i} + 5E_{2i} + 3E_{4i}) \quad i, j = 1, NP$$

3.5 For fifth order element (NP=6)

$$E_{0i} = \frac{1}{768}(9\delta_{1i} - 75\delta_{2i} + 450\delta_{3i} + 450\delta_{4i} - 75\delta_{5i} + 9\delta_{6i})$$

$$E_{1i} = \frac{1}{768}(-9\delta_{1i} - 125\delta_{2i} - 2250\delta_{3i} + 2250\delta_{4i} - 125\delta_{5i} + 9\delta_{6i})$$

$$E_{2i} = \frac{1}{768}(-250\delta_{1i} + 1950\delta_{2i} - 1700\delta_{3i} - 1700\delta_{4i} + 1950\delta_{5i} - 250\delta_{6i})$$

$$E_{3i} = \frac{1}{768}(250\delta_{1i} - 3250\delta_{2i} + 8500\delta_{3i} - 8500\delta_{4i} + 3250\delta_{5i} - 250\delta_{6i})$$

$$E_{4i} = \frac{1}{768}(625\delta_{1i} - 1875\delta_{2i} + 1250\delta_{3i} + 1250\delta_{4i} - 1875\delta_{5i} + 625\delta_{6i})$$

$$E_{5i} = \frac{1}{768}(-625\delta_{1i} + 3125\delta_{2i} - 6250\delta_{3i} + 6250\delta_{4i} - 3125\delta_{5i} + 625\delta_{6i})$$

$$K_{ij}^{G} = \frac{L}{3465}[231E_{0j}(15E_{0i} + 5E_{2i} + 3E_{4i}) + 33E_{1j}(35E_{1i} + 21E_{3i} + 15E_{5i}) + 33E_{2j}(35E_{0i} + 21E_{2i} + 15E_{4i}) + 11E_{3j}(63E_{1i} + 45E_{3i} + 35E_{5i}) + 11E_{4j}(63E_{0i} + 45E_{2i} + 35E_{4i}) + 5E_{5j}(99E_{1i} + 77E_{3i} + 63E_{5i})]$$

$$K_{ij}^{D} = \frac{4}{315L}[315E_{1j}(E_{1i} + E_{3i} + E_{5i}) + 85E_{2j}(5E_{2i} + 6E_{4i}) + 9E_{3j}(35E_{1i} + 63E_{3i} + 75E_{5i}) + 72E_{4j}(7E_{2i} + 10E_{4i}) + 5E_{5j}(63E_{1i} + 135E_{3i} + 175E_{5i})]$$

$$F_i = \frac{L}{15}(15E_{0i} + 5E_{2i} + 3E_{4i}) \quad i, j = 1, NP$$

3.6 For sixth order element (NP=7)

$$E_{0i} = \delta_{4i},$$

$$E_{1i} = \frac{1}{80}(-4\delta_{1i} + 36\delta_{2i} - 180\delta_{3i} + 180\delta_{5i} - 36\delta_{6i} + 4\delta_{7i})$$

$$E_{2i} = \frac{1}{80}(4\delta_{1i} - 54\delta_{2i} + 540\delta_{3i} - 980\delta_{4i} + 540\delta_{5i} - 54\delta_{6i} + 4\delta_{7i})$$

$$E_{3i} = \frac{1}{80}(45\delta_{1i} - 360\delta_{2i} + 585\delta_{3i} - 586\delta_{5i} + 360\delta_{6i} - 45\delta_{7i})$$

$$E_{4i} = \frac{1}{80}(-45\delta_{1i} + 360\delta_{2i} - 1755\delta_{3i} + 2520\delta_{4i} - 1755\delta_{5i} + 360\delta_{6i} - 45\delta_{7i})$$

$$E_{5i} = \frac{1}{80}(-81\delta_{1i} + 324\delta_{2i} - 81\delta_{3i} + 405\delta_{5i} - 324\delta_{6i} + 81\delta_{7i})$$

$$E_{6i} = \frac{1}{80}(81\delta_{1i} - 486\delta_{2i} + 1215\delta_{3i} - 1620\delta_{4i} + 1215\delta_{5i} - 486\delta_{6i} + 81\delta_{7i})$$

$$K_{ij}^{G} = \frac{L}{45045}[429E_{0j}(105E_{0i} + 35E_{2i} + 21E_{4i} + 15E_{6i}) + 429E_{1j}(35E_{1i} + 21E_{3i} + 15E_{5i}) + 143E_{2j}(105E_{0i} + 63E_{2i} + 45E_{4i} + 35E_{6i}) + 143E_{3j}(63E_{1i} + 45E_{3i} + 35E_{5i}) + 13E_{4j}(693E_{0i} + 495E_{2i} + 385E_{4i} + 315E_{6i}) + 65E_{5j}(99E_{1i} + 77E_{3i} + 63E_{5i}) + 5E_{6j}(1287E_{0i} + 1001E_{2i} + 819E_{4i} + 693E_{6i})]$$

$$K_{ij}^{D} = \frac{4}{3465L}[3465E_{1j}(E_{1i} + E_{3i} + E_{5i}) + 132E_{2j}(35E_{2i} + 42E_{4i} + 45E_{6i}) + 99E_{3j}(35E_{1i} + 42E_{3i} + 45E_{5i}) + 264E_{4j}(35E_{2i} + 30E_{4i} + 35E_{6i}) + 55E_{5j}(63E_{1i} + 135E_{3i} + 175E_{5i}) + 60E_{6j}(99E_{2i} + 154E_{4i} + 189E_{6i})]$$

$$F_i = \frac{L}{105}(105E_{0i} + 35E_{2i} + 21E_{4i} + 15E_{6i}) \quad i, j = 1, NP$$

3.7 For seventh order element (NP=8)

$$E_{0i} = \frac{1}{92160}(-225\delta_{1i} + 2205\delta_{2i} - 11025\delta_{3i} + 55125\delta_{4i} + 55125\delta_{5i} - 11025\delta_{6i} + 2205\delta_{7i} - 225E_{8i})$$

$$E_{1i} = \frac{1}{92160}(225\delta_{1i} - 3087\delta_{2i} - 25725\delta_{3i} - 385875\delta_{4i} + 385875\delta_{5i} + 25725\delta_{6i} + 3087\delta_{7i} - 225E_{8i})$$

$$E_{2i} = \frac{1}{92160}(12691\delta_{1i} - 122255\delta_{2i} + 572859\delta_{3i} - 463295\delta_{4i} - 463295\delta_{5i} + 572859\delta_{6i} - 122255\delta_{7i} + 12691E_{8i})$$

$$E_{3i} = \frac{1}{92160}(-12691\delta_{1i} + 171157\delta_{2i} - 1336671\delta_{3i} + 3243065\delta_{4i} - 3243065\delta_{5i} + 1336671\delta_{6i} - 171157\delta_{7i} + 12691E_{8i})$$

$$E_{4i} = \frac{1}{92160}(-84035\delta_{1i} + 708295\delta_{2i} - 1620675\delta_{3i} + 996415\delta_{4i} + 996415\delta_{5i} - 1620675\delta_{6i} + 708295\delta_{7i} - 84035E_{8i})$$

$$E_{5i} = \frac{1}{92160}(84035\delta_{1i} - 991613\delta_{2i} + 3781575\delta_{3i} - 6974904\delta_{4i} + 6974904\delta_{5i} - 3781575\delta_{6i} + 991613\delta_{7i} - 84035E_{8i})$$

$$E_{6i} = \frac{1}{92160}(117649\delta_{1i} - 588245\delta_{2i} + 1058841\delta_{3i} - 588245\delta_{4i} - 588245\delta_{5i} + 1058841\delta_{6i} - 588245\delta_{7i} + 117649E_{8i})$$

$$E_{7i} = \frac{1}{92160}(-117649\delta_{1i} + 823543\delta_{2i} - 2470629\delta_{3i} + 4117715\delta_{4i} - 4117715\delta_{5i} + 2470629\delta_{6i} - 823543\delta_{7i} + 117649E_{8i})$$

$$K_{ij}^{G} = \frac{L}{45045}[429E_{0j}(105E_{0i} + 35E_{2i} + 21E_{4i} + 15E_{6i}) + 143E_{1j}(105E_{1i} + 63E_{3i} + 45E_{5i} + 35E_{7i}) + 143E_{2j}(105E_{0i} + 63E_{2i} + 45E_{4i} + 35E_{6i}) + 13E_{3j}(693E_{1i} + 495E_{3i} + 385E_{5i} + 315E_{7i}) + 13E_{4j}(693E_{0i} + 495E_{2i} + 385E_{4i} + 315E_{6i}) + 5E_{5j}(1287E_{1i} + 1001E_{3i} + 819E_{5i} + 693E_{7i}) + 5E_{6j}(1287E_{0i} + 1001E_{2i} + 819E_{4i} + 693E_{6i}) + 7E_{7j}(715E_{1i} + 585E_{3i} + 495E_{5i} + 429E_{7i})]$$

$$K_{ij}^{D} = \frac{4}{45045L}[45045E_{1j}(E_{1i} + E_{3i} + E_{5i} + E_{7i}) + 1716E_{2j}(35E_{2i} + 42E_{4i} + 45E_{6i}) + 143E_{3j}(315E_{1i} + 567E_{3i} + 675E_{5i} + 735E_{7i}) + 1144E_{4j}(63E_{2i} + 90E_{4i} + 105E_{6i}) + 65E_{5j}(693E_{1i} + 1485E_{3i} + 1925E_{5i} + 2205E_{7i}) + 780E_{6j}(99E_{2i} + 154E_{4i} + 189E_{6i}) + 35E_{7j}(1287E_{1i} + 3549E_{3i} + 4095E_{5i} + 4851E_{7i})]$$

$$F_i = \frac{L}{105}(105E_{0i} + 35E_{2i} + 21E_{4i} + 15E_{6i}) \qquad i, j = 1, NP$$

3.8 Eight order element (NP = 9)

$$E_{0i} = \delta_{5i},$$

$$E_{1i} = \frac{1}{210}(3\delta_{1i} - 32\delta_{2i} + 168\delta_{3i} - 672\delta_{4i} + 672\delta_{6i} - 168\delta_{7i} + 32\delta_{8i} - 3\delta_{9i})$$

$$E_{2i} = \frac{1}{630}(-9\delta_{1i} + 128\delta_{2i} - 1008\delta_{3i} + 8064\delta_{4i} - 14350\delta_{5i} + 8064\delta_{6i} - 1008\delta_{7i} + 128\delta_{8i} - 9\delta_{9i})$$

$$E_{3i} = \frac{1}{45}(-14\delta_{1i} + 9\delta_{2i} - 676\delta_{3i} + 976\delta_{4i} - 976\delta_{6i} + 676\delta_{7i} - 9\delta_{8i} + 14\delta_{9i})$$

$$E_{4i}=\frac{1}{45}(14\delta_{1i}-192\delta_{2i}+1352\delta_{3i}-3904\delta_{4i}+5460\delta_{5i}-3904\delta_{6i}+1352\delta_{7i}-192\delta_{8i}+14\delta_{9i})$$

$$E_{5i}=\frac{1}{45}(64\delta_{1i}-576\delta_{2i}+1664\delta_{3i}-1856\delta_{4i}+1856\delta_{6i}-1664\delta_{7i}+576\delta_{8i}-64\delta_{9i})$$

$$E_{6i}=\frac{1}{45}(-64\delta_{1i}+768\delta_{2i}-3328\delta_{3i}+7424\delta_{4i}-9600\delta_{5i}+7424\delta_{6i}-3328\delta_{7i}+768\delta_{8i}-64\delta_{9i})$$

$$E_{7i}=\frac{1}{315}(-512\delta_{1i}+3072\delta_{2i}-7168\delta_{3i}+7168\delta_{4i}-7168\delta_{6i}+7168\delta_{7i}-3072\delta_{8i}+512\delta_{9i})$$

$$E_{8i}=\frac{1}{315}(512\delta_{1i}-4096\delta_{2i}+14336\delta_{3i}-28672\delta_{4i}+35840\delta_{5i}-28672\delta_{6i}+14336\delta_{7i}-4096\delta_{8i}+512\delta_{9i})$$

$$K_{ij}^{G}=\frac{L}{765765}[2431E_{0j}(315E_{0i}+105E_{2i}+63E_{4i}+45E_{6i}+35E_{8i})+2431E_{1j}(105E_{1i}+63E_{3i}+45E_{5i}+35E_{7i})+221E_{2j}(1155E_{0i}+693E_{2i}+495E_{4i}+385E_{6i}+315E_{8i})+221E_{3j}(693E_{1i}+495E_{3i}+385E_{5i}+315E_{7i})+17E_{4j}(9009E_{0i}+6435E_{2i}+5005E_{4i}+4095E_{6i}+3465E_{8i})+85E_{5j}(1287E_{1i}+1001E_{3i}+819E_{5i}+693E_{7i})+17E_{6j}(6435E_{0i}+5005E_{2i}+4095E_{4i}+3465E_{6i}+3003E_{8i})+119E_{7j}(715E_{1i}+585E_{3i}+495E_{5i}+429E_{7i})+7E_{8j}(12155E_{0i}+9945E_{2i}+8415E_{4i}+7293E_{6i}+6435E_{8i})]$$

$$K_{ij}^{D}=\frac{4}{45045L}[45045E_{ij}(E_{1i}+E_{3i}+E_{5i}+E_{7i})+572E_{2j}(105E_{2i}+126E_{4i}+135E_{6i}+140E_{8i})+143E_{3j}(315E_{1i}+567E_{3i}+675E_{5i}+735E_{7i})+104E_{4j}(693E_{2i}+995E_{4i}+1155E_{6i}+1260E_{8i})+65E_{5j}(693E_{1i}+1485E_{3i}+1925E_{5i}+2205E_{7i})+60E_{6j}(1287E_{2i}+2002E_{4i}+2457E_{6i}+2772E_{8i})+35E_{7j}(1287E_{1i}+3003E_{3i}+4095E_{5i}+4851E_{7i})+112E_{8j}(715E_{2i}+1170E_{4i}+1485E_{6i}+1716E_{8i})]$$

$$F_{i}=\frac{L}{315}(315E_{0i}+105E_{2i}+63E_{4i}+45E_{6i}+35E_{8i})\qquad i,j=1,NP$$

3.9 Ninth order element (NP = 10)

$$E_{0i}=\frac{1}{2293760}(1225\delta_{1i}-14175\delta_{2i}+79380\delta_{3i}-308700\delta_{4i}+1389150\delta_{5i}+1389150\delta_{6i}-308700\delta_{7i}+79380\delta_{8i}-14175\delta_{9i}+1225\delta_{10i})$$

$$E_{1i}=\frac{1}{2293760}(-1225\delta_{1i}+18225\delta_{2i}-142884\delta_{3i}+926100\delta_{4i}-12502350\delta_{5i}+12502350\delta_{6i}-926100\delta_{7i}+142884\delta_{8i}-18225\delta_{9i}+1225\delta_{10i})$$

$$E_{2i}=\frac{1}{2293760}(-116244\delta_{1i}+1335852\delta_{2i}-7354800\delta_{3i}+26823860\delta_{4i}-20688696\delta_{5i}-20688696\delta_{6i}+26823860\delta_{7i}-7354800\delta_{8i}+1335852\delta_{9i}-116244\delta_{10i})$$

$$E_{3i}=\frac{1}{2293760}(116244\delta_{1i}-1717524\delta_{2i}+13238640\delta_{3i}-80471664\delta_{4i}+186198264\delta_{5i}-186198264\delta_{6i}+80471664\delta_{7i}-13238640\delta_{8i}+1717524\delta_{9i}-116244\delta_{10i})$$

$$E_{4i}=\frac{1}{2293760}(1439046\delta_{1i}-15788682\delta_{2i}+76953240\delta_{3i}-150518088\delta_{4i}+87914484\delta_{5i}+87914484\delta_{6i}-150518088\delta_{7i}+76953240\delta_{8i}-15788682\delta_{9i}+1439046\delta_{10i})$$

$$E_{5i}=\frac{1}{2293760}(-1439046\delta_{1i}+20299734\delta_{2i}-138515832\delta_{3i}+451554264\delta_{4i}-791230356\delta_{5i}+791230356\delta_{6i}-451554264\delta_{7i}+138515832\delta_{8i}-20299734\delta_{9i}+1439046\delta_{10i})$$

$$E_{6i}=\frac{1}{2293760}(-4960116\delta_{1i}+47947788\delta_{2i}-165337200\delta_{3i}+257926032\delta_{4i}-135576504\delta_{5i}-135576504\delta_{6i}+257926032\delta_{7i}-165337200\delta_{8i}+47947788\delta_{9i}-4960116\delta_{10i})$$

$$E_{7i}=\frac{1}{2293760}(4960116\delta_{1i}-61647156\delta_{2i}+297606960\delta_{3i}-773778096\delta_{4i}+1220188536\delta_{5i}-1220188536\delta_{6i}+773778096\delta_{7i}-297606960\delta_{8i}+61647156\delta_{9i}-4960116\delta_{10i})$$

$$E_{8i}=\frac{1}{2293760}(4782969\delta_{1i}-33480783\delta_{2i}+95659380\delta_{3i}-133923132\delta_{4i}+66961566\delta_{5i}+66961566\delta_{6i}-133923132\delta_{7i}+95659380\delta_{8i}-33480783\delta_{9i}+4782969\delta_{10i})$$

$$E_{9i}=\frac{1}{2293760}(-4782969\delta_{1i}+43046721\delta_{2i} -172186884\delta_{3i}+401769396\delta_{4i} -602654096\delta_{5i}+602654096\delta_{6i} -401769396\delta_{7i}+172186884\delta_{8i} -43046721\delta_{9i}+4782969\delta_{10i})$$

$$K_{ij}^{G}=\frac{L}{14549535}[46189E_{0j}(315E_{0i}+105E_{2i}+63E_{4i} +45E_{6i}+35E_{8i})+4199E_{1j}(1155E_{1i}+693E_{3i} +495E_{5i}+385E_{7i}+315E_{9i})+4199E_{2j}(1155E_{0i} +693E_{2i}+495E_{4i}+385E_{6i}+315E_{8i})+323E_{3j} (9009E_{1i}+6435E_{3i}+5005E_{5i}+4095E_{7i} +3465E_{9i})+323E_{4j}(9009E_{0i}+6435E_{2i} +5005E_{4i}+4095E_{6i}+3465E_{8i})+323E_{5j} (6435E_{1i}+5005E_{3i}+4095E_{5i}+3465E_{7i} +3003E_{9i})+323E_{6j}(6435E_{0i}+5005E_{2i} +4095E_{4i}+3465E_{6i}+3003E_{8i})+133E_{7j} (12155E_{1i}+9945E_{3i}+8415E_{5i}+7293E_{7i} +6435E_{9i})+133E_{8j}(12155E_{0i}+9945E_{2i} +8415E_{4i}+7293E_{6i}+6435E_{8i})+21E_{9j} (62985E_{1i}+53295E_{3i}+46189E_{5i}+40755E_{7i} +36465E_{9i})]$$

$$K_{ij}^{D}=\frac{4}{765765L}[765765E_{1j}(E_{1i}+E_{3i}+E_{5i}+E_{7i}+E_{9i}) +9724E_{2j}(105E_{2i}+126E_{4i}+135E_{6i}+140E_{8i}) +221E_{3j}(3465E_{1i}+6237E_{3i}+7425E_{5i}+8085E_{7i} +8505E_{9i})+1768E_{4j}(693E_{2i}+990E_{4i}+1155E_{6i} +1260E_{8i})+85E_{5j}(9009E_{1i}+19305E_{3i} +25025E_{5i}+28665E_{7i}+31185E_{9i})+1020E_{6j} (1287E_{2i}+2002E_{4i}+2457E_{6i}+2772E_{8i}) +119E_{7j}(6435E_{1i}+15015E_{3i}+20475E_{5i} +24255E_{7i}+27027E_{9i})+1904E_{8j}(715E_{2i} +1170E_{4i}+1485E_{6i}+1716E_{8i})+63E_{9j}(12155E_{1i} +29835E_{3i}+42075E_{5i}+51051E_{7i}+57915E_{9i})]$$

$$F_{i}=\frac{L}{315}(315E_{0i}+105E_{2i}+63E_{4i}+45E_{6i}+35E_{8i})$$

$$i,j=1,NP$$

and similarly

3.10 For tenth order element (NP=11)

$$E_{0j}=\delta_{6i}$$

$$E_{1j}=\frac{1}{145152}(-578\delta_{1i}+7200\delta_{2i}-43200\delta_{3i} +172800\delta_{4i}-604800\delta_{5i}+604800\delta_{7i} -17200\delta_{8i}+43200\delta_{9i}-7200\delta_{10i}+576\delta_{11i})$$

$$E_{2j}=\frac{1}{145152}(578\delta_{1i}-9000\delta_{2i}+7200\delta_{3i} -432000\delta_{4i}+3024000\delta_{5i}-5311152\delta_{6i} +3024000\delta_{7i}-432000\delta_{8i}+72000\delta_{9i} -9000\delta_{10i}+576\delta_{11i})$$

$$E_{3j}=\frac{1}{145152}(20500\delta_{1i}-252200\delta_{2i}+1468800\delta_{3i} -5242800\delta_{4i}+7009800\delta_{5i}-7009800\delta_{7i} +5242800\delta_{8i}-1468800\delta_{9i}+252200\delta_{10i} -20500\delta_{11i})$$

$$E_{4j}=\frac{1}{145152}(-20500\delta_{1i}+315250\delta_{2i}-2434500\delta_{3i} +13107000\delta_{4i}-35049000\delta_{5i}+48163500\delta_{6i} -35049000\delta_{7i}+13107000\delta_{8i}-2434500\delta_{9i} +315250\delta_{10i}-20500\delta_{11i})$$

$$E_{5j}=\frac{1}{145152}(-170625\delta_{1i}+1995000\delta_{2i}-10276875\delta_{3i} +24570000\delta_{4i}-25436250\delta_{5i}+25436250\delta_{7i} -24570000\delta_{8i}+10276875\delta_{9i}-1995000\delta_{10i} +170625\delta_{11i})$$

$$E_{6j}=\frac{1}{145152}(170625\delta_{1i}-249374\delta_{2i}+17128125\delta_{3i} -61425000\delta_{4i}+127181250\delta_{5i}-161122500\delta_{6i} +127181250\delta_{7i}-61425000\delta_{8i}+17128125\delta_{9i} -249374\delta_{10i}+170625\delta_{11i})$$

$$E_{7j}=\frac{1}{145152}(467850\delta_{1i}-4875000\delta_{2i}+19406250\delta_{3i} -38250000\delta_{4i}+35437500\delta_{5i}-35437500\delta_{7i} +38250000\delta_{8i}-19406250\delta_{9i}+4875000\delta_{10i} -467850\delta_{11i})$$

$$E_{8j}=\frac{1}{145152}(-467850\delta_{1i}+643750\delta_{2i}-32343750\delta_{3i} +95625000\delta_{4i}-35437500\delta_{5i}+216562500\delta_{6i} -35437500\delta_{7i}+95625000\delta_{8i}-32343750\delta_{9i} +643750\delta_{10i}-467850\delta_{11i})$$

$$E_{9j}=\frac{1}{145152}(-390625\delta_{1i}+3125000\delta_{2i} -10546875\delta_{3i}+18750000\delta_{4i} -163406250\delta_{5i}+163406250\delta_{7i} -18750000\delta_{8i}+10546875\delta_{9i} -13125000\delta_{10i}+390625\delta_{11i})$$

$$E_{10j}=\frac{390625}{145152}(\delta_{1i}-10\delta_{2i}+45\delta_{3i}-120\delta_{4i}+210\delta_{5i} -252\delta_{6i}+210\delta_{7i}-120\delta_{8i}+45\delta_{9i}-10\delta_{10i}+\delta_{11i})$$

$$
\begin{aligned}
K_{ij}^{G} = \frac{\mathrm{L}}{101846745}[& 29393E_{0j}(3465E_{0i}+1155E_{2i} \\
& +693E_{4i}+495E_{6i}+385E_{8i}+315E_{10i}) \\
& +29393E_{1j}(1155E_{1i}+693E_{3i}+495E_{5i} \\
& +385E_{7i}+315E_{9i})+2261E_{2j}(15015E_{0i} \\
& +9009E_{2i}+6535E_{4i}+5005E_{6i}+4095E_{8i} \\
& +3465E_{10i})+2261E_{3j}(9009E_{1i}+6435E_{3i} \\
& +5005E_{5i}+4095E_{7i}+3465E_{9i})+2261E_{4j} \\
& (9009E_{0i}+6435E_{2i}+5005E_{4i}+4095E_{6i} \\
& +3465E_{8i}+3003E_{10i})+2261E_{5j}(6435E_{1i} \\
& +5005E_{3i}+4095E_{5i}+3465E_{7i}+3003E_{9i}) \\
& +133E_{6j}(109395E_{0i}+85085E_{2i}+69615E_{4i} \\
& +58905E_{6i}+51051E_{8i}+45045E_{10i})+931E_{7j} \\
& (12155E_{1i}+9945E_{3i}+8415E_{5i}+7293E_{7i} \\
& +6435E_{9i})+49E_{8j}(230945E_{0i}+188955E_{2i} \\
& +159885E_{4i}+138567E_{6i}+122265E_{8i} \\
& +109395E_{10i})+147E_{9j}(62985E_{1i}+53295E_{3i} \\
& +46189E_{5i}+40755E_{7i}+36465E_{9i})+21E_{10j} \\
& (440895E_{0i}+373065E_{2i}+323323E_{4i} \\
& +285285E_{6i}+255255E_{8i}+230945E_{10i})]
\end{aligned}
$$

$$
\begin{aligned}
K_{ij}^{D} = \frac{4}{14549535L}[& 14549535E_{1j}(E_{1i}+E_{3i}+E_{5i}+E_{7i} \\
& +E_{9i})+16796E_{2j}(1155E_{2i}+1386E_{4i}+1485E_{6i} \\
& +1540E_{8i}+1575E_{10i})+4199E_{3j}(3465E_{1i} \\
& +6237E_{3i}+7425E_{5i}+8085E_{7i}+8505E_{9i}) \\
& +2584E_{4j}(9009E_{2i}+12870E_{4i}+15015E_{6i} \\
& +16380E_{8i}+17325E_{10i})+1615E_{5j}(9009E_{1i} \\
& +19305E_{3i}+25025E_{5i}+28665E_{7i}+31185E_{9i}) \\
& +3876E_{6j}(6435E_{2i}+10010E_{4i}+12285E_{6i} \\
& +13860E_{8i}+15015E_{10i})+2261E_{7j}(6435E_{1i} \\
& +15015E_{3i}+20475E_{5i}+24255E_{7i}+27027E_{9i}) \\
& +2128E_{8j}(12155E_{2i}+19890E_{4i}+25245E_{6i} \\
& +29172E_{8i}+32175E_{10i})+399E_{9j}(36465E_{1i} \\
& +89505E_{3i}+126225E_{5i}+15315E_{7i} \\
& +173745E_{9i})+420E_{10j}(62985E_{2i}+106590E_{4i} \\
& +138567E_{6i}+163020E_{8i}+182325E_{10i})]
\end{aligned}
$$

$$
\begin{aligned}
F_i = \frac{L}{3465}(& 3465E_{0i}+1155E_{2i}+696E_{4i}+495E_{6i} \\
& +385E_{8i}+315E_{10i}) \qquad i, j = 1, NP
\end{aligned}
$$

4. Application Examples

Example-1: If $D = 1, G = 0, Q = -2$, a = 1, b = 3 and $\alpha_0 = 1, \alpha_1 = 0, \beta_0 = 0, \beta_1 = 1, \gamma_1 = 1$ and $\gamma_2 = 6$ then the Eq.(1-2) reduces to Poisson equation. Considering the domain as one quadratic element as in Fig.-1, we obtain

$u_2 = u(2) = 4$ and $u_3 = u(3) = 9$

which are accurate comparing with the analytical solution $u = x^2$

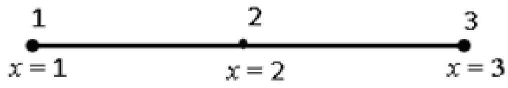

Fig.-1

Example-2:

If $D = k_x A, G = -hP, Q = hPu_s$, a = 0, b = 5 and

$\alpha_0 = 1, \alpha_1 = 0, \beta_0 = 0, \beta_1 = 1, \gamma_1 = 140$ and $\gamma_2 = 0$

then the Eq.(1-2) reduces to one-dimensional heat transfer equation. Considering the domain as (i) two linear element, (ii) one quadratic element, (iii) two quadratic elements, (iv) one cubic element, (v) one fourth order element and (vi) one fifth order element solutions are obtained and compared with the exact solution

$$u = 100(\cosh\frac{\sqrt{7}x}{7} - \tanh\frac{5\sqrt{7}}{7}\sinh\frac{\sqrt{7}x}{7}) + 40$$

Obtained solutions in cases (iii)-(vi) are in full agreement with the exact solution. Several physical problems are considered and in every cases higher order finite element formulation produced satisfactory results, which are not included for the page limit. One can use the appended software for the solution of any physical problem deduced by the one-dimensional equilibrium problem.

5. Conclusion

In this paper finite element formulation of one-dimensional equilibrium problem employing linear (1st order) to tenth order elements are presented. Expressions for the element matrices are presented in compact form and that helped to write the computer code. The code based on the present formulation is tested through the application examples. In all the problems, it is found that the higher order element formulation produced exact solutions. Thus, we believe that the user can use the developed software considering any order up to tenth of the element to get the satisfactory results for the field problems under consideration. The software needs only the input for the order specification of the element used in discretizations, the end points of the element(s) and the values of the parameters used in the general boundary conditions. The developed software is interactive, so the user will understand easily for the necessary inputs comparing with the general equilibrium problem. The beauty of the present formulation and the software is that if the elements of same length are used then element matrices will be evaluated at once with no repetitions. But for the different lengths of the elements, corresponding element matrices will be formed individually for each one.

As it is recommended that if the function variations occurs frequently in some portions of the boundary of the domain then at that portion of the domain higher order elements are preferable for the descretization [1-7]. Otherwise many lower order elements are required and that substantially increase the computing process to get the satisfactory results. But, some times lower order element formulation is also preferable to mesh the domain. Considering all the facts formulations and a suitable code are developed in this paper. So, the analyst can use the code as his choice for the demand. The necessity, usefulness and the performance of the code easily can be justified by [1, 7-10].

Thus, we believe that the presented software based on the element formulation in appendix will help all the finite element users to solve one-dimensional field problems, which are generally described by the One-dimensional equilibrium problem using any kind of computer.

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Appendix

```
C PROGRAM P1DEM.FOR (PROGRAM FOR ONE
DIMENSIONALBOUNDARY VALUE PROBLEM)
C TO  DETERMINE THE ELEMENT MATRICES FOR
LOWER TO HIGHER
C ORDER ELEMENTS, NE= NUMBER  OF
ELEMENTS, NP=NUMBER OF POINTS  PER
ELEMENT
C E.G., NP=2 FOR FIRST ORDER,NP=3,4,5,..,11
C FOR SECOND, 3-RD,  4-TH, 5-TH, 6-TH, 7-TH,
8-TH, 9-TH AND 10-TH ORDER RESPECTIVELY,
C THESE ARE THE MODIFICATION IN C
PARAMETER STATE MENT.
      PARAMETER(NE=1,NP=5)
      CHARACTER*8,OUTFILE
      DOUBLE PRECISION
X(NP),F(NP),KD(NP,NP),KG(NP,NP),B(NP),
     1
E(0:NP-1,NP),D1(NP,NP),XF,XL,EL,AL0,AL1,BTA0,B
TA1,
     2  AA,BB,DUA,DUB,THETA1(NP),THETA2(NP)
      PRINT*,'TYPE THE FILE NAME FOR OUTPUT
OF ELEMENT MATRICES'
      READ*,OUTFILE
      OPEN( UNIT =3, FILE= OUTFILE)
      PRINT*,'TYPE THE THE VALUE OF A AND B
THAT IS DOMAIN'
      READ*,AA,BB
 PRINT*,'TYPE THE VALUES OF
AL0,AL1,BTA0,BTA1
     1         MATCHING WITH THE BOUNDARY
CONDITIONS'
      READ*,AL0,AL1,BTA0,BTA1
      PRINT*,'TYPE THE VALUES OF DUA,DUB'
      READ*,DUA,DUB
C**FORMATION OF D1 MATRIX ,KRONEKER
DELTA TO DETERMINE MATRICES A AND B**
```

```
      DO 5 IC = 1, NP
   DO 5 JC = 1, NP
     IF(IC.EQ.JC) THEN

      D1(IC,JC) =1.D0
     ELSE
      D1(IC,JC) = 0.D0
     ENDIF
5    CONTINUE

IF((AL1.NE.0.0).AND.(BTA1.NE.0.0))THEN
     ICOND=1
     WRITE(*,6)
6    FORMAT(3X,'BOTH BOUNDARY
CONDITIONS ARE SUPPRESSIBLE')
    ELSE
IF((AL1.NE.0.0).AND.(BTA1.EQ.0.0)) THEN
     ICOND=2
     WRITE(*,7)
7    FORMAT(3X,'BOUNDARY
CONDITIONS SUPPRESSIBLE AT X=A',/3X,
  1   'ESSENTIAL AT X=B')
    ELSE
IF((AL1.EQ.0.0).AND.(BTA1.NE.0.0)) THEN
     ICOND=3
     WRITE(*,8)
8    FORMAT(3X,'ESSENTIAL BOUNDARY
CONDITION AT X=A',/3X,
  1   'SUPPRESSIBLE BOUNDARY
CONDITION AT X=B')

 ELSE
IF((AL1.EQ.0.0).AND.(BTA1.EQ.0.0)) THEN
     ICOND=4
     WRITE(*,9)
9    FORMAT(3X,'ESSENTIAL BOUNDARY
CONDITIONS AT X=A AND AT X=B')

    END IF
C*****************************************

   DO 14 N1 = 1,NE
   WRITE(*,10) N1
10  FORMAT('TYPE TWO END POINTS XF
AND XL FOR ELEMENT:- ',I2,/)

   READ*,XF,XL
 EL=XL-XF
   CALL EMAT(NP,EL,D1,E,F)
   CALL ELEMAT(NP,EL,E,KG,KD)
C
   DO 15 II=1,NP
     IF(ICOND.EQ.1) THEN
       IF((II.EQ.1).AND.(XF.EQ.AA)) THEN
        THETA1(II)=1.D0
       ELSE
         THETA1(II)=0.D0
       END IF
       IF((II.EQ.NP).AND.(XL.EQ.BB)) THEN
        THETA2(II)=1.D0
       ELSE
        THETA2(II)=0.D0
       END IF
      B(II)=THETA2(II)*DUB-
THETA1(II)*DUA
     END IF
     IF(ICOND.EQ.2) THEN
       IF((II.EQ.1).AND.(XF.EQ.AA)) THEN
        THETA1(II)=1.D0
       ELSE
        THETA1(II)=0.D0
       END IF
      B(II)=THETA1(II)*DUA
     END IF

 IF(ICOND.EQ.3) THEN
      IF((II.EQ.NP).AND.(XL.EQ.BB)) THEN
        THETA2(II)=1.D0
       ELSE
        THETA2(II)=0.D0
       END IF
      B(II)=THETA2(II)*DUB
     END IF
     IF(ICOND.EQ.4) B(II)=0.D0
 15  CONTINUE
C ========================================
```

```
      WRITE (3,18) N1
 18   FORMAT(/,3X, 'GIVEN END POINTS
FOR ELEMENT:-',I2,/)
      WRITE( 3,19) XF,XL
 19   FORMAT(3X,'XF=',D18.10,3X,'XL=',D18.10,/3X,
'AND COMPUTED VALUES ARE:',/)
      WRITE(3,20) (IL,F(IL),IL=1,NP)
 20   FORMAT(3X,'F(',I2,')=',D18.10)
      WRITE(3,25)
 25
FORMAT(3X,'=========******============ ')
      WRITE(3,30) (IP,B(IP),IP=1,NP)
 30   FORMAT(3X,'B(',I2,')=',D18.10)
      WRITE(3,35)
 35
FORMAT(3X,'==========******============ ')
C*****THIS PART PRINTS THE COMPONENTS OF
ELEMENT MATRICES*******
      DO 40 I =1,NP
      DO 40 J =1,NP
      WRITE(3,50) I,J,KD(I,J),I,J,KG(I,J)
 50   FORMAT(3X,'KD(',I2,',',I2,')=', D18.10,3X,
'KG(',I2,',',I2,')=',D18.10)
 40   CONTINUE
      WRITE(3,55)
 55
FORMAT(3X,'===========****===========',/)

14    CONTINUE
      PRINT*,'YOUR DATA FILE IS  ', OUTFILE
      STOP
      END
C======================================
      SUBROUTINE EMAT(NP,EL,D1,E,F)
      DOUBLE PRECISION
D1(NP,NP),E(0:NP-1,NP),EL,F(NP)
      DO 70 I = 1,NP
       IF(NP.EQ.2) THEN
        E(0,I)= (D1(1,I)+D1(2,I))/2.D0
        E(1,I)= (D1(2,I)-D1(1,I))/2.D0
        F(I) = EL*E(0,I)
      ELSE IF(NP.EQ.3) THEN
        E(0,I)=D1(2,I)
        E(1,I)= (D1(3,I)-D1(1,I))/2.D0
        E(2,I)= (D1(1,I)-2*D1(2,I)+D1(3,I))/2.D0
        F(I) = EL*(3*E(0,I)+E(2,I))/3.D0
      ELSE IF(NP.EQ.4) THEN
        E(0,I)=
(-D1(1,I)+9*D1(2,I)+9*D1(3,I)-D1(4,I))/16.D0
        E(1,I)=
(D1(1,I)-27*D1(2,I)+27*D1(3,I)-D1(4,I))/16.D0
        E(2,I)=9*(D1(1,I)-D1(2,I)-
D1(3,I)+D1(4,I))/16.D0

E(3,I)=9*(-D1(1,I)+3*D1(2,I)-3*D1(3,I)+D1(4,I))/16.D
0
        F(I) = EL*(3*E(0,I)+E(2,I))/3.D0
      ELSE IF(NP.EQ.5) THEN
        E(0,I)=D1(3,I)
        E(1,I)=
(D1(1,I)-8*D1(2,I)+8*D1(4,I)-D1(5,I))/6.D0
        E(2,I)=
(-D1(1,I)+16*D1(2,I)-30*D1(3,I)+16*D1(4,I)
-D1(5,I))/6.D0

E(3,I)=4*(-D1(1,I)+2*D1(2,I)-2*D1(4,I)+D1(5,I))/6.D0
        E(4,I)=4*(D1(1,I)-4*D1(2,I)+6*D1(3,I)-
4*D1(4,I) +D1(5,I))/6.D0
        F(I) =
EL*(15*E(0,I)+5*E(2,I)+3*E(4,I))/15.D0
      ELSE IF(NP.EQ.6) THEN

E(0,I)=(9*D1(1,I)-75*D1(2,I)+450*D1(3,I)+450*D1(4,I
)-75*D1(5,I)+9*D1(6,I))/768.D0

E(1,I)=(-9*D1(1,I)-125*D1(2,I)-2250*D1(3,I)+2250*D1
(4,I) -125*D1(5,I)+9*D1(6,I))/768.D0

E(2,I)=(-250*D1(1,I)+1950*D1(2,I)-1700*D1(3,I)-1700
```

```
     1         +3250*D1(5,I)-250*D1(6,I))/768.D0

E(4,I)=(625*D1(1,I)-1875*D1(2,I)+1250*D1(3,I)+1250
*D1(4,I)
     1         -1875*D1(5,I)+625*D1(6,I))/768.D0

E(5,I)=(-625*D1(1,I)+3125*D1(2,I)-6250*D1(3,I)+6250
*D1(4,I)
     1         -3125*D1(5,I)+625*D1(6,I))/768.D0
        F(I) =
EL*(15*E(0,I)+5*E(2,I)+3*E(4,I))/15.D0
       ELSE IF(NP.EQ.7) THEN
        E(0,I)=D1(4,I)

E(1,I)=(-4*D1(1,I)+36*D1(2,I)-180*D1(3,I)+180*D1(5,
I) -36*D1(6,I)+4*D1(7,I))/80.D0

E(2,I)=(4*D1(1,I)-54*D1(2,I)+540*D1(3,I)-980*D1(4,I)
     1         +540*D1(5,I)-
54*D1(6,I)+4*D1(7,I))/80.D0
        E(3,I)=(45*D1(1,I)-
360*D1(2,I)+585*D1(3,I)-585*D1(5,I)
+360*D1(6,I)-45*D1(7,I))/80.D0

E(4,I)=(-45*D1(1,I)+360*D1(2,I)-1755*D1(3,I)+2520*
D1(4,I)
     1
-1755*D1(5,I)+360*D1(6,I)-45*D1(7,I))/80.D0

E(5,I)=(-81*D1(1,I)+324*D1(2,I)-81*D1(3,I)+405*D1(
5,I)
     1         -324*D1(6,I)+81*D1(7,I))/80.D0
        E(6,I)=(81*D1(1,I)-
486*D1(2,I)+1215*D1(3,I)-1620*D1(4,I)
     1
+1215*D1(5,I)-486*D1(6,I)+81*D1(7,I))/80.D0
        F(I) =
EL*(105*E(0,I)+35*E(2,I)+21*E(4,I)+15*E(6,I))/
105.D0
  ELSE IF(NP.EQ.8) THEN

E(0,I)=(-225*D1(1,I)+2205*D1(2,I)-11025*D1(3,I)+551
25*D1(4,I)
     1
+385875*D1(5,I)+25725*D1(6,I)+3087*D1(7,I)
-225*D1(8,I))/92160.D0

E(2,I)=(12691*D1(1,I)-122255*D1(2,I)+572859*D1(3,I
) -
463295*D1(4,I)-463295*D1(5,I)
     1         +572859*D1(6,I)
-122255*D1(7,I)+12691*D1(8,I))/92160.D0

E(3,I)=(-12691*D1(1,I)+171157*D1(2,I)-1336671*D1(3
,I) +3243065*D1(4,I)-
3243065*D1(5,I)
     1         +1336671*D1(6,I)
-171157*D1(7,I)+12691*D1(8,I))/92160.D0

E(4,I)=(-84035*D1(1,I)+708295*D1(2,I)-1620675*D1(3
,I)
+996415*D1(4,I)+996415*D1(5,I)
     1
-1620675*D1(6,I)+708295*D1(7,I)-84035*D1(8,I))/921
60.D0

E(5,I)=(84035*D1(1,I)-991613*D1(2,I)+3781575*D1(3,
I) -6974904*D1(4,I)+6974904*D1(5,I)
     1
-3781575*D1(6,I)+991613*D1(7,I)-84035*D1(8,I))/921
60.D0

E(6,I)=(117649*D1(1,I)-588245*D1(2,I)+1058841*D1(
3,I) -
588245*D1(4,I)-588245*D1(5,I)
     1
+1058841*D1(6,I)-588245*D1(7,I)+117649*D1(8,I))/92
160.D0

E(7,I)=(-117649*D1(1,I)+823543*D1(2,I)-2470629*D1(
3,I) +4117715*D1(4,I)-
4117715*D1(5,I)
     1
+2470629*D1(6,I)-823543*D1(7,I)+117649*D1(8,I))/92
160.D0
        F(I) =
EL*(105*E(0,I)+35*E(2,I)+21*E(4,I)+15*E(6,I))/
105.D0
```

```
E(3,I)=(-14*D1(1,I)+9*D1(2,I)-676*D1(3,I)+976*D1(4,I)
    1
-976*D1(6,I)+676*D1(7,I)-9*D1(8,I)+14*D1(9,I))/45.D0
      E(4,I)=(14*D1(1,I)-192*D1(2,I)+1352*D1(3,I)-3904*D1(4,I)
+5460*D1(5,I)-3904*D1(6,I)
    1
+1352*D1(7,I)-192*D1(8,I)+14*D1(9,I))/45.D0
      E(5,I)=(64*D1(1,I)-576*D1(2,I)+1664*D1(3,I)-1856*D1(4,I)
    1       +1856*D1(6,I)-1664*D1(7,I)+576*D1(8,I)-64*D1(9,I))/45.D0

E(6,I)=(-64*D1(1,I)+768*D1(2,I)-3328*D1(3,I)+7424*D1(4,I) -9600*D1(5,I)
    1       +7424*D1(6,I)-3328*D1(7,I)+768*D1(8,I)-64*D1(9,I))/45.D0

E(7,I)=(-512*D1(1,I)+3072*D1(2,I)-7168*D1(3,I)+7168*D1(4,I) -7168*D1(6,I)
    1
+7168*D1(7,I)-3072*D1(8,I)+512*D1(9,I))/315.D0

E(8,I)=(512*D1(1,I)-4096*D1(2,I)+14336*D1(3,I)-28672*D1(4,I) +35840*D1(5,I)
    1
-28672*D1(6,I)+14336*D1(7,I)-4096*D1(8,I)+512*D1(9,I))/315.D0
      F(I) =
EL*(315*E(0,I)+105*E(2,I)+63*E(4,I)+45*E(6,I)+35*E(8,I))/315.D0
      ELSE IF(NP.EQ.10) THEN

E(0,I)=(1225*D1(1,I)-14175*D1(2,I)+79380*D1(3,I)
-308700*D1(4,I)+1389150*D1(5,I) +1389150
    1       *D1(6,I)-308700*D1(7,I)+79380*D1(8,I)-14175*D1(9,I)
+1225*D1(10,I))/2293760.D0

E(1,I)=(-1225*D1(1,I)+18225*D1(2,I)-142884*D1(3,I)+926100*D1(4,I)-
2502350*D1(5,I)
    2
+1335852*D1(9,I)-116244*D1(10,I))/2293760.D0

E(3,I)=(116244*D1(1,I)-1717524*D1(2,I)+13238640*D1(3,I) -80471664*D1(4,I)
    1       +186198264*D1(5,I)-186198264*D1(6,I)
+80471664*D1(7,I)-13238640*D1(8,I)
    2
+1717524*D1(9,I)-116244*D1(10,I))/2293760.D0

E(4,I)=(1439046*D1(1,I)-15788682*D1(2,I)+76953240*D1(3,I) -150518088*D1(4,I)
    1       +87914484*D1(5,I)+87914484*D1(6,I)
-150518088*D1(7,I)+76953240*D1(8,I)
    2
-15788682*D1(9,I)+1439046*D1(10,I))/2293760.D0
      E(5,I)=(-1439046*D1(1,I)+20299734*D1(2,I)-138515832*D1(3,I) +451554264*D1(4,I)
    1       -791230356*D1(5,I)+791230356*D1(6,I)
-451554264*D1(7,I)+138515832*D1(8,I)
    2
-20299734*D1(9,I)+1439046*D1(10,I))/2293760.D0
      E(6,I)=(-4960116*D1(1,I)+47947788*D1(2,I)-165337200*D1(3,I) +257926032*D1(4,I)
    1       -135576504*D1(5,I)-135576504*D1(6,I)
+257926032*D1(7,I)-165337200*D1(8,I)
    2
+47947788*D1(9,I)-4960116*D1(10,I))/2293760.D0

E(7,I)=(4960116*D1(1,I)-61647156*D1(2,I)+297606960*D1(3,I) -773778096*D1(4,I)
    1       +1220188536*D1(5,I)
-1220188536*D1(6,I)+773778096*D1(7,I)
    2       -297606960*D1(8,I)+61647156*D1(9,I)-4960116*D1(10,I))/2293760.D0

E(8,I)=(4782969*D1(1,I)-33480783*D1(2,I)+95659380*D1(3,I) -
```

```
133923132*D1(4,I)
    1         +66961566*D1(5,I)
+66961566*D1(6,I)-133923132*D1(7,I)+95659380*D1(
8,I)
    2
-33480783*D1(9,I)+4782969*D1(10,I))/2293760.D0
       E(9,I)=(-
4782969*D1(1,I)+43046721*D1(2,I)-172186884*D1(3,I
) +401769396*D1(4,I)
    1         -
602654096*D1(5,I)+602654096*D1(6,I)-401769396*D
1(7,I)+172186884*D1(8,I)
    3
-43046721*D1(9,I)+4782969*D1(10,I))/2293760.D0
       F(I) =
EL*(315*E(0,I)+105*E(2,I)+63*E(4,I)+45*E(6,I)+
35*E(8,I))/315.D0
       ELSE IF(NP.EQ.11) THEN
       E(0,I)=D1(6,I)

E(1,I)=(-578*D1(1,I)+7200*D1(2,I)-43200*D1(3,I)+172
800*D1(4,I) -604800*D1(5,I) +604800
    1         *D1(7,I)-
172800*D1(8,I)+43200*D1(9,I)-7200*D1(10,I)+576*D
1(11,I))/145152.D0
       E(2,I)=(578*D1(1,I)-
9000*D1(2,I)+7200*D1(3,I)-432000*D1(4,I)
+3024000*D1(5,I)-5311152
    1
*D1(6,I)+3024000*D1(7,I)-432000*D1(8,I)+72000*D1(
9,I)-9000*D1(10,I)
    2         +576*D1(11,I))/145152.D0

E(3,I)=(20500*D1(1,I)-252200*D1(2,I)+1468800*D1(3,
I) -5242800*D1(4,I)+7009800*D1(5,I)
    1        -7009800*D1(7,I)
+5242800*D1(8,I)-1468800*D1(9,I)+252200*D1(10,I)
    2        -20500*D1(11,I))/145152.D0

E(4,I)=(-20500*D1(1,I)+315250*D1(2,I)-2434500*D1(3
,I) +13107000*D1(4,I)-35049000*D1(5,I)
    1
+48163500*D1(6,I)-35049000*D1(7,I)+13107000*D1(8
,I)-
2434500*D1(9,I)
    3
24570000*D1(8,I)+10276875*D1(9,I)-1995000*D1(10,I
)
    3         +170625*D1(11,I))/145152.D0

E(6,I)=(170625*D1(1,I)-249374*D1(2,I)+17128125*D1
(3,I) -61425000*D1(4,I)+127181250
    1
*D1(5,I)-161122500*D1(6,I)+127181250*D1(7,I)-6142
5000*D1(8,I)+17128125*D1(9,I)
    2
-249374*D1(10,I)+170625*D1(11,I))/145152.D0

E(7,I)=(468750*D1(1,I)-4875000*D1(2,I)+19406250*D
1(3,I) -38250000*D1(4,I)+35437500
    1
*D1(5,I)-35437500*D1(7,I)+38250000*D1(8,I)-194062
50*D1(9,I)+4875000*D1(10,I)
    2         -468750*D1(11,I))/145152.D0

E(8,I)=(-468750*D1(1,I)+643750*D1(2,I)-32343750*D
1(3,I) +95625000*D1(4,I)-35437500
    1
*D1(5,I)+216562500*D1(6,I)-35437500*D1(7,I)+95625
000*D1(8,I)-32343750*D1(9,I)
    2
+643750*D1(10,I)-468750*D1(11,I))/145152.D0

E(9,I)=(-390625*D1(1,I)+3125000*D1(2,I)-10546875*
D1(3,I) +18750000*D1(4,I)-163406250
    1
*D1(5,I)+163406250*D1(7,I)-18750000*D1(8,I)+10546
875*D1(9,I)-3125000*D1(10,I)
    2         +390625*D1(11,I))/145152.D0
       E(10,I)=(390625*(D1(1,I)-
10*D1(2,I)+45*D1(3,I)-120*D1(4,I)
+210*D1(5,I)-252*D1(6,I)+210
    1
*D1(7,I)-120*D1(8,I)+45*D1(9,I)-10*D1(10,I)+D1(11,I
)))/145152.D0
       F(I) =
EL*(3465*E(0,I)+1155*E(2,I)+696*E(4,I)+495*E
(6,I)+385*E(8,I)+315*E(10,I))/3465.D0
       END IF
 70   CONTINUE
```

```
1,NP),KG(NP,NP),KD(NP,NP)
      DO 75 I = 1,NP
      DO 75 J = 1,NP
      IF(NP.EQ.2) THEN
KG(I,J)=EL*(3*E(0,I)*E(0,J)+E(1,I)*E(1,J))/3.D0
       KD(I,J)=(4/EL)*E(1,I)*E(1,J)
       ELSE IF(NP.EQ.3) THEN
KG(I,J)=EL*(15*E(0,I)*E(0,J)+5*E(2,I)*E(0,J)+5
*E(1,I)*E(1,J)
    1
+5*E(0,I)*E(2,J)+3*E(2,J)*E(2,I))/15.D0
KD(I,J)=(4/EL)*(3*E(1,I)*E(1,J)+4*E(2,J)*E(2,I))/3.D0
       ELSE IF(NP.EQ.4) THEN
KG(I,J)=EL*(105*E(0,I)*E(0,J)+35*E(2,I)*E(0,J)
+35*E(1,I)*E(1,J) +21*E(1,J)*E(3,I)+35*E(2,J)
    1
*E(0,I)+21*E(2,J)*E(2,I)+21*E(3,J)*E(1,I)+15*E
(3,I)*E(3,J))/105.D0

KD(I,J)=(4/EL)*(15*E(1,I)*E(1,J)+15*E(1,J)*E(3,
I)+20*E(2,J)
    1
*E(2,I)+15*E(3,J)*E(1,I)+27*E(3,J)*E(3,I))/15.D0
       ELSE IF(NP.EQ.5) THEN
KG(I,J)=EL*(21*E(0,J)*(15*E(0,I)+5*E(2,I)+3*E(4,I))
+21*E(1,J)*(5*E(1,I)+3*E(3,I))
    1
+3*E(2,J)*(35*E(0,I)+21*E(2,I)+15*E(4,I))+9*E
(3,J)*(7*E(1,I)+5*E(3,I))
    2
+E(4,J)*(63*E(0,I)+45*E(2,I)+35*E(4,I)))/315.D0
KD(I,J)=(4/EL)*(105*E(1,J)*(E(1,I)+E(3,I))+28*E(2,J)*
(5*E(2,I) +6*E(4,I))+21*E(3,J)*(5*E(1,I)
    1
+9*E(3,I))+24*E(4,J)*(7*E(2,I)+10*E(4,I)))/105.D0
       ELSE IF(NP.EQ.6) THEN
KG(I,J)=EL*(231*E(0,J)*(15*E(0,I)+5*E(2,I)+3*E(4,I))
+33*E(1,J) *(35*E(1,I)+21*E(3,I)
    1
+15*E(5,I))+33*E(2,J)*(35*E(0,I)+21*E(2,I)+15*E(4,I
)+11*E(3,J)*(63*E(1,I)+45*E(3,I)
    2       +35
*E(5,I))+11*E(4,J)*(63*E(0,I)+45*E(2,I)+35*E(4,I))+5
    3
*E(5,J)*(99*E(1,I)+77*E(3,I)+63*E(5,I)))/3465.D0
KD(I,J)=(4/EL)*(315*E(1,J)*(E(1,I)+E(3,I)+E(5,I)
)+84*E(2,J)
    1
*(5*E(2,I)+6*E(4,I))+9*E(3,J)*(35*E(1,I)+63*E(3,I)
+75
    2
*E(5,I))+72*E(4,J)*(7*E(2,I)+10*E(4,I))+5*E(5,J)*(63
    3
*E(1,I)+135*E(3,I)+175*E(5,I)))/315.D0
       ELSE IF(NP.EQ.7) THEN
KG(I,J)=EL*(429*E(0,J)*(105*E(0,I)+35*E(2,I)+21*
E(4,I) +15*E(6,I))+429*E(1,J)*(35*E(1,I)
    1
+21*E(3,I)+15*E(5,I))+143*E(2,J)*(105*E(0,I)+63
*E(2,I)+45*E(4,I)+35*E(6,I)) +143*E(3,J)
    2
*(63*E(1,I)+45*E(3,I)+35*E(5,I))+13*E(4,J)*(693
*E(0,I)+495*E(2,I)+385*E(4,I)
    3
+315*E(6,I))+65*E(5,J)*(99*E(1,I)+77*E(3,I)+63
*E(5,I)) +5*E(6,J)*(1287*E(0,I)+1001
    4
*E(2,I)+819*E(4,I)+693*E(6,I)))/45045.D0
KD(I,J)=(4/EL)*(3465*E(1,J)*(E(1,I)+E(3,I)+E(5,
I))+132*E(2,J) *(35*E(2,I)+42*E(4,I)+45*E(6,I))
    1
+99*E(3,J)*(35*E(1,I)+42*E(3,I)+45*E(5,I))+264
*E(4,J)*(35*E(2,I)+30*E(4,I) +35*E(6,I))
    2
+55*E(5,J)*(63*E(1,I)+135*E(3,I)+175*E(5,I))
+60*E(6,J)*(99*E(2,I)+154*E(4,I)
    3       +189*E(6,I)))/3465.D0
       ELSE IF(NP.EQ.8) THEN
KG(I,J)=EL*(429*E(0,J)*(105*E(0,I)+35*E(2,I)+
21*E(4,I)+15*E(6,I))+143*E(1,J)*(105*E(1,I)
    1
+63*E(3,I)+45*E(5,I)+35*E(7,I))+143*E(2,J)*(105
*E(0,I)+63*E(2,I)+45*E(4,I)+35*E(6,I))
    2
+13*E(3,J)*(693*E(1,I)+495*E(3,I)+385*E(5,I)+315
*E(7,I))+13*E(4,J)*(693*E(0,I)+495
    3        *E(2,I)
+385*E(4,I)+315*E(6,I))+5*E(5,J)*(1287*E(1,I)+
1001*E(3,I)+819*E(5,I) +693
```

```
    4
*E(7,I))+5*E(6,J)*(1287*E(0,I)+1001*E(2,I)+819
*E(4,I)+693*E(6,I))+7*E(7,J)*(715*E(1,I)

    5
+585*E(3,I)+495*E(5,I)+429*E(7,I)))/45045.D0

KD(I,J)=(4/EL)*(45045*E(1,J)*(E(1,I)+E(3,I)+E
(5,I)+E(7,I))+1716*E(2,J)*(35*E(2,I)+42*E(4,I)

    1
+45*E(6,I))+143*E(3,J)*(315*E(1,I)+567*E(3,I)+
675*E(5,I)+735+E(7,I))+1144*E(4,J)

    3
*(63*E(2,I)+90*E(4,I)+105*E(6,I))+65*E(5,J)*(693
*E(1,I) +1485*E(3,I)+1925*E(5,I)+2205

    4
*E(7,I))+780*E(6,J)*(99*E(2,I)+154*E(4,I)+189*
E(6,I))+35*E(7,J)*(1287

    5
*E(1,I)+3549*E(3,I)+4095*E(5,I)+4851*E(7,I)))/
45045.D0
      ELSE IF(NP.EQ.9) THEN

KG(I,J)=EL*(2431*E(0,J)*(315*E(0,I)+105*E(2,I)
+63*E(4,I) +45*E(6,I)+35*E(8,I))+2431*E(1,J)
    1
*(105*E(1,I)+63*E(3,I)+45*E(5,I)+35*E(7,I))+
221*E(2,J)*(1155*E(0,I)+693*E(2,I) +495

    2
*E(4,I)+385*E(6,I)+315*E(8,I))+221*E(3,J)
*(693*E(1,I)+495*E(3,I)+385*E(5,I)+315*E(7,I))

    3
+17*E(4,J)*(9009*E(0,I)+6435*E(2,I)+5005*E(4,I)
+4095*E(6,I) +3465*E(8,I))+85*E(5,J)

    4
*(1287*E(1,I)+1001*E(3,I)+819*E(5,I)+693
*E(7,I))+17*E(6,J)*(6435*E(0,I) +5005*E(2,I)

    5
+4095*E(4,I)+3465*E(6,I)+3003*E(8,I))
+119*E(7,J)*(715*E(1,I)+585*E(3,I) +495*E(5,I)

    6
+429*E(7,I))+7*E(8,J)*(12155*E(0,I)+9945*E(2,I)
+8415*E(4,I)+7293*E(6,I)

    7        +6435*E(8,I)))/765765.D0

KD(I,J)=(4/EL)*(45045*E(1,J)*(E(1,I)+E(3,I)+E(5,I)
+E(7,I)) +572*E(2,J)*(105*E(2,I)+126*E(4,I)

    1
+135*E(6,I)+140*E(8,I))+143*E(3,J)*(315*E(1,I)
+567*E(3,I) +675*E(5,I)+735*E(7,I))+104

    2
*E(4,J)*(693*E(2,I)+995*E(4,I)+1155*E(6,I)+1260
*E(8,I))+65*E(5,J) *(693*E(1,I)+1485

    3
*E(3,I)+1925*E(5,I)+2205*E(7,I))+60*E(6,J)*(1287*
E(2,I)+2002*E(4,I)+2457*E(6,I) +2772

    4
*E(8,I))+35*E(7,J)*(1287*E(1,I)+3003*E(3,I)+4095*
E(5,I)+4851*E(7,I))+112*E(8,J)

    5
*(715*E(2,I)+1170*E(4,I)+1485*E(6,I)+1716
*E(8,I)))/45045.D0
      ELSE IF(NP.EQ.10) THEN

KG(I,J)=EL*(46189*E(0,J)*(315*E(0,I)+105*E
(2,I)+63*E(4,I) +45*E(6,I)+35*E(8,I)) +4199

    1
*E(1,J)*(1155*E(1,I)+693*E(3,I)+495*E(5,I)+385
*E(7,I)+315*E(9,I))+4199*E(2,J)

    3
*(1155*E(0,I)+693*E(2,I)+495*E(4,I)+385*E(6,I)
+315*E(8,I))+323*E(3,J) *(9009*E(1,I)

    4
+6435*E(3,I)+5005*E(5,I)+4095*E(7,I)+3465*E
(9,I))+323*E(4,J)*(9009*E(0,I)+6435*E(2,I)

    6
+5005*E(4,I)+4095*E(6,I)+3465*E(8,I))
+323*E(5,J)*(6435*E(1,I)+5005*E(3,I)
+4095*E(5,I)

    7
+3465*E(7,I)+3003*E(9,I))+323*E(6,J)*(6435*E
(0,I) +5005*E(2,I)+4095*E(4,I)+3465*E(6,I)

    9
+3003*E(8,I))+133*E(7,J)*(12155*E(1,I)+9945*E
(3,I)+8415*E(5,I)+7293*E(7,I)+6435

    2
*E(9,I))+133*E(8,J)*(12155*E(0,I)+9945*E(2,I)+
8415*E(4,I)+7293*E(6,I)+6435*E(8,I))

    3
+21*E(9,J)*(62985*E(1,I)+53295*E(3,I)+46189*E(5,I)
+40755*E(7,I)

    5        +36465*E(9,I)))/14549535.D0

KD(I,J)=(4/EL)*(765765*E(1,J)*(E(1,I)+E(3,I)+E(5,I)+
E(7,I) +E(9,I))+9724*E(2,J)*(105*E(2,I)

    1
+126*E(4,I)+135*E(6,I)+140*E(8,I))+221*E(3,J)*
(3465*E(1,I)+6237*E(3,I)+7425*E(5,I)

    2
+8085*E(7,I)+8505*E(9,I))+1768*E(4,J)*(693*E
(2,I) +990*E(4,I)+1155*E(6,I)+1260*E(8,I))
```

```
3
+85*E(5,J)*(9009*E(1,I)+19305*E(3,I)+25025*E
(5,I)+28665*E(7,I) +31185*E(9,I))+1020

4
*E(6,J)*(1287*E(2,I)+2002*E(4,I)+2457*E(6,I)+
2772*E(8,I))+119*E(7,J)*(6435*E(1,I)

5
+15015*E(3,I)+20475*E(5,I)+24255*E(7,I)+27027
*E(9,I)) +1904*E(8,J)*(715*E(2,I)+1170

6
*E(4,I)+1485*E(6,I)+1716*E(8,I))+63*E(9,J)*
(12155*E(1,I)+29835*E(3,I)

7
+42075*E(5,I)+51051*E(7,I)+57915*E(9,I))
)/765765.D0

    ELSE IF(NP.EQ.11) THEN

KG(I,J)=EL*(29393*E(0,J)*(3465*E(0,I)+1155*E

(2,I)+693*E(4,I) +495*E(6,I)+385*E(8,I)+315

1
*E(10,I))+29393*E(1,J)*(1155*E(1,I)+693*E(3,I)
+495*E(5,I)+385*E(7,I)+315*E(9,I)) +2261

2
*E(2,J)*(15015*E(0,I)+9009*E(2,I)+6435*E(4,I)+
5005*E(6,I)+4095*E(8,I)+3465*E(10,I))

3
+2261*E(3,J)*(9009*E(1,I)+6435*E(3,I)+5005*E
(5,I)+4095*E(7,I)+3465*E(9,I)) +2261

4
*E(4,J)*(9009*E(0,I)+6435*E(2,I)+5005*E(4,I)+4095*
E(6,I)+3465*E(8,I)+3003*E(10,I))

5
+2261*E(5,J)*(6435*E(1,I)+5005*E(3,I)+4095*E
(5,I)+3465*E(7,I)+3003*E(9,I)) +133*E(6,J)

6
*(109395*E(0,I)+85085*E(2,I)+69615*E(4,I)
+58905*E(6,I)+51051*E(8,I)+45045*E(10,I))

7
+931*E(7,J)*(12155*E(1,I)+9945*E(3,I)+8413
*E(5,I)+7293*E(7,I)+6435*E(9,I))+49*E(8,J)

8
*(230945*E(0,I)+188955*E(2,I)+159885*E(4,I)+
138567*E(6,I)+122265*E(8,I)+109395

9
*E(10,I))+147*E(9,J)*(62985*E(1,I)+53295*E(3,I)
+46189*E(5,I)+40755*E(7,I)+36465

1
*E(9,I))+21*E(10,J)*(440895*E(0,I)+373065*E(2,I)
+323323*E(4,I)+285285*E(6,I)

2
+255255*E(8,I)+230945*E(10,I)))/101846745.D0

KD(I,J)=(4/EL)*(14549535*E(1,J)*(E(1,I)+E(3,I)
+E(5,I)+E(7,I) +E(9,I))+16796*E(2,J)*(1155

1
*E(2,I)+1386*E(4,I)+1485*E(6,I)+1540*E(8,I)+
1575*E(10,I))+4199*E(3,J)*(3465*E(1,I)

3
+6237*E(3,I)+7425*E(5,I)+8085*E(7,I)+8505
*E(9,I))+2584*E(4,J)*(9009*E(2,I) +12870

4
*E(4,I)+15015*E(6,I)+16380*E(8,I)+17325*E(10,I))
+1615*E(5,J)*(9009*E(1,I)+19305*E(3,I)

6
+25025*E(5,I)+28665*E(7,I)+31185*E(9,I))+3876
*E(6,J)*(6435*E(2,I)+10010*E(4,I) +12285

8
*E(6,I)+13860*E(8,I)+15015*E(10,I))+2261*E
(7,J)*(6435*E(1,I)+15015*E(3,I)+20475*E(5,I)

9
+24255*E(7,I)+27027*E(9,I))+2128*E(8,J)*(12155
*E(2,I)+19890*E(4,I)+25245*E(6,I)

2
+29172*E(8,I)+32175*E(10,I))+399*E(9,J)*(36465
*E(1,I)+89505*E(3,I)+126225*E(5,I)

3
+15315*E(7,I)+173745*E(9,I))+420*E(10,J)*(62985
*E(2,I)+106590*E(4,I)+138567

5
*E(6,I)+163020*E(8,I)+182325*E(10,I)))/14549535.D0

    END IF

75  CONTINUE

    RETURN

    END
```