SUST Journal of Science and Technology Vol 35(1) 2025, 16-22

SUST Journal of Science and Technology

journal homepage: https://journals.sust.edu

Modification of Heisenberg's Uncertainty Relation for a Damped Harmonic Oscillator under the Generalized Uncertainty Principle

P. Karmaker[1], S. B. Faruque[1]

Department of Physics, Shahjalal University of Science and Technology, Sylhet – 3114, Bangladesh

Article Info

Keywords:

Modification
Operators
Uncertainty Product
Time evolution
Damped Harmonic
GUP

Abstract

We investigate the effects of the Generalized Uncertainty Principle (GUP), which intro- duces a fundamental minimal observable length, on the quantum dynamics of a damped harmonic oscillator. By incorporating a modified Heisenberg algebra into the framework, we derive the altered uncertainty relations for damped quantum systems and examine their dependence on the discrete energy spectrum. Our analysis reveals significant devia- tions from the standard Heisenberg uncertainty relation, demonstrating that the presence of a minimal length scale fundamentally modifies the quantum-classical correspondence in dissipative systems. We present explicit expressions for the modified uncertainty relations and discuss their physical implications for quantum measurements in damped oscillatory systems. These results contribute to our understanding of quantum mechanics at length scales approaching the Planck regime and provide testable predictions for experimental verification of GUP effects in dissipative quantum systems.

1. Introduction

The quantum harmonic oscillator is a fundamental system that forms the basis for under- standing a wide range of physical phenomena. When dissipation is introduced [1], the system is described as a damped harmonic oscillator, which requires careful treatment in the quantum domain due to the non-conservative nature of the dynamics. The correspond- ing uncertainty relations follow directly from the canonical commutation relations [2], which are defined by the Heisenberg algebra.

Recent developments in quantum gravity and related approaches indicate that the Heisenberg algebra may be modified at very small length scales [5].

2. Operator Formalism of the Simple Harmonic Os- cillator

The Hamiltonian of the harmonic oscillator can be written as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

where the spring constant has been expressed as $k = m\omega^2$ and m is the mass of oscillating particle and ω is the angular frequency. One may feel the tempting urge to factorize this Hamiltonian and write it as [4]

$$\frac{1}{2}m\omega^2\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \tag{2}$$

but a problem immediately arises. In quantum mechanics, the operators $\hat{x}$ and $\hat{p}$ do not commute. Consequently, multiplying out the brackets in the above expression does not reproduce $\hat{H}$ exactly, but instead yields

$$\frac{1}{2}m\omega^2\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) = \frac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{2m} + \frac{i\omega}{2}[\hat{x}, \hat{p}] \tag{3}$$

where

$$[\hat{x}, \hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \tag{4}$$

is the canonical commutator of the position and momentum operators. The right-hand side of Eq. (3) is therefore equal to $\hat{H} - \hbar\omega/2$, which differs from $\hat{H}$ by the subtraction of the zero-point energy.

The factorization can thus be made consistent, and we observe that the operators

$$\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \quad \text{and} \quad \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

* Corresponding author.

E-mail addresses: parthokarmaker13@gmail.com (P. Karmaker)

https://doi.org/10.63512/sustjst.2025.1004

Received 23 October 2025; Received in revised form 3 April 2026; Accepted 15 April 2026

play a central role. Since $\hat{x}$ and $\hat{p}$ are Hermitian operators, these two operators are adjoints of each other and are therefore not themselves Hermitian. As a result, they do not correspond to physical observables.

We introduce the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, defined with a normalization factor chosen for later convenience:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \tag{5}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \tag{6}$$

These definitions can be inverted to express the position and momentum operators in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \tag{7}$$

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right) \tag{8}$$

3. Ladder Operators for the Damped Harmonic Oscillator

We now consider the case of the damped harmonic oscillator. The Hamiltonian corresponding to damped harmonic motion is the Caldirola–Kanai (CK) Hamiltonian [1], which is given by

$$\hat{H} = \frac{\hat{p}^2}{2m}e^{-\gamma t} + \frac{1}{2}m\omega^2\hat{x}^2 e^{\gamma t}. \tag{9}$$

Where, γ is the damping constant. As in the case of the simple harmonic oscillator, the CK Hamiltonian can be factorized as

$$\frac{1}{2}m\omega^2 e^{\gamma t}\left(\hat{x} - \frac{ie^{-\gamma t}}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{ie^{-\gamma t}}{m\omega}\hat{p}\right) \tag{10}$$

which leads to

$$\begin{aligned}\frac{1}{2}m\omega^2 e^{\gamma t}&\left(\hat{x} - \frac{ie^{-\gamma t}}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{ie^{-\gamma t}}{m\omega}\hat{p}\right)\\ &= \frac{\hat{p}^2}{2m}e^{-\gamma t} + \frac{1}{2}m\omega^2\hat{x}^2 e^{\gamma t} + \frac{i\omega}{2}[\hat{x},\hat{p}]\end{aligned} \tag{11}$$

In complete analogy with the simple harmonic oscillator, we can now define the annihilation and creation operators for the damped harmonic oscillator as

$$\hat{a}(t) = \sqrt{\frac{m\omega}{2\hbar}}\left(e^{\gamma t/2}\hat{x} + \frac{i}{m\omega}e^{-\gamma t/2}\hat{p}\right) \tag{12}$$

$$\hat{a}^\dagger(t) = \sqrt{\frac{m\omega}{2\hbar}}\left(e^{\gamma t/2}\hat{x} - \frac{i}{m\omega}e^{-\gamma t/2}\hat{p}\right) \tag{13}$$

The ladder operators for the damped harmonic oscillator explicitly contain exponential time-dependent factors. This time dependence arises from the Caldirola–Kanai Hamiltonian, which itself is explicitly time dependent. Consequently, the position and momentum operators are effectively modified by the damping factor. In the next section, we shall determine the corresponding operators when the Generalized Uncertainty Principle is applied.

4. Introduction to GUP in the Quantum Operators for Harmonic Motion

Usually, the commutation relation between the quantum operators $q_{k\alpha}$ and $p_{k\alpha}$ is given by [3]

$$[q_{k\alpha}, p_{k'\alpha'}] = i\hbar\,\delta_{kk'}\delta_{\alpha\alpha'} \tag{14}$$

We now assume the presence of a Generalized Uncertainty Principle (GUP) that introduces a minimal observable length, but no minimal observable momentum. In this case, the modified commutation relation takes the form

$$[q_{k\alpha}, p_{k'\alpha'}] = i\hbar\,\delta_{kk'}\delta_{\alpha\alpha'}\left(\mathbb{I} + \beta p_{k\alpha}^2\right) \tag{15}$$

where $\mathbb{I}$ denotes the identity operator and β is a positive constant related to the existence of a minimal observable length.

For a single degree of freedom, the above relation reduces to the simpler form

$$[\hat{x},\hat{p}] = i\hbar\left(1 + \beta\hat{p}^2\right) \tag{16}$$

This modified commutation relation leads directly to the generalized Heisenberg uncertainty relation

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}\left[1 + \beta(\Delta p)^2\right] \tag{17}$$

where β is assumed to be positive and independent of Δx and Δp. The above inequality implies the existence of a minimum measurable length determined by the parameter β.

Such generalized uncertainty relations have appeared naturally in several approaches to quantum gravity and string theory, based on the premise that a fundamental minimal length should be described quantum mechanically as a minimal uncertainty in position measurements.

At this stage, an important question arises: how should one define the Fock space in the presence of a GUP? By definition, the Fock space construction relies on creation and annihilation operators. However, in this modified framework, the usual definitions of these operators in terms of the position and momentum operators are no longer valid, since they do not reproduce the modified commutation relations.

We therefore consider a generalized form of the annihilation and creation operators:

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} + i\left[\hat{p} + f(\hat{p})\right]\right) \tag{18}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} - i\left[\hat{p} + f(\hat{p})\right]\right) \tag{19}$$

where $f(\hat{p})$ is an unknown function of the momentum operator. This function is required to satisfy certain conditions, which will be specified in the following discussion. The function $f(\hat{p})$ introduced above is required to satisfy the following conditions:

(i) In the limit $\beta \to 0$, the usual definitions of the creation and annihilation operators are recovered.

(ii) For $\beta \neq 0$, the operators satisfy the modified commutation relation given in Eq. (5).

(iii) The ladder operators obey the algebra

$$[\hat{a}_{k\alpha}, \hat{a}^{\dagger}_{k'\alpha'}] = i\hbar\, \delta_{kk'}\delta_{\alpha\alpha'} \tag{20}$$

It can be readily shown that the unique function satisfying all of the above requirements is [3]

$$f(\hat{p}) = \sum_{n=1}^{\infty} \frac{(-\beta)^n}{2n+1}\, \hat{p}^{\,2n+1} \tag{21}$$

Condition (iii) ensures that the usual results associated with the Fock space structure remain valid in the present framework. In particular, the number operator can be defined as $\hat{N} = \hat{a}^{\dagger}\hat{a}$, and the operators $\hat{a}^{\dagger}$ and $\hat{a}$ retain their interpretations as creation and annihilation operators, respectively.

Using the explicit form of $f(\hat{p})$, the annihilation and creation operators may be rewritten in a compact form as

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} + \frac{i}{\sqrt{\beta}}\tan^{-1}\!\left(\sqrt{\beta}\,\hat{p}\right)\right) \tag{22}$$

$$\hat{a}^{\dagger} = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} - \frac{i}{\sqrt{\beta}}\tan^{-1}\!\left(\sqrt{\beta}\,\hat{p}\right)\right) \tag{23}$$

Here, the operator-valued function $(\sqrt{\beta})^{-1}\tan^{-1}(\sqrt{\beta}\,\hat{p})$ represents the series expansion

$$\hat{p} + \sum_{n=1}^{\infty} \frac{(-\beta)^n}{2n+1}\, \hat{p}^{\,2n+1} \tag{24}$$

The position and momentum operators can also be expressed explicitly in terms of the ladder operators as

$$\hat{p} = \frac{1}{\sqrt{\beta}}\tan\left[-i\sqrt{\frac{\hbar m\omega\beta}{2}}\,(\hat{a} - \hat{a}^{\dagger})\right] \tag{25}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^{\dagger}) \tag{26}$$

5. GUP-Modified Operators for the Damped Harmonic Oscillator

During the construction of the ladder operators $\hat{a}$ and $\hat{a}^{\dagger}$ for the damped harmonic oscillator, we observed that, in comparison with the simple harmonic oscillator, the position operator $\hat{x}$ is modified by a factor $e^{\gamma t/2}$, while the momentum operator $\hat{p}$ is modified by a factor $e^{-\gamma t/2}$.

In an analogous manner, we may now define the ladder operators for the damped harmonic oscillator in the presence of the Generalized Uncertainty Principle (GUP) as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(e^{\gamma t/2}\hat{x} + \frac{i\,e^{-\gamma t/2}}{m\omega\sqrt{\beta}}\tan^{-1}\!\left(\sqrt{\beta}\,\hat{p}\right)\right) \tag{27}$$

$$\hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(e^{\gamma t/2}\hat{x} - \frac{i\,e^{-\gamma t/2}}{m\omega\sqrt{\beta}}\tan^{-1}\!\left(\sqrt{\beta}\,\hat{p}\right)\right) \tag{28}$$

Subtracting Eqs. (24) and (25), we obtain

$$\hat{a} - \hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{2i\,e^{-\gamma t/2}}{m\omega\sqrt{\beta}}\tan^{-1}\!\left(\sqrt{\beta}\,\hat{p}\right)\right) \tag{29}$$

which leads to the momentum operator

$$\hat{p} = \frac{1}{\sqrt{\beta}}\tan\left[-i\sqrt{\frac{\hbar m\omega\beta}{2}}\,e^{\gamma t/2}\,(\hat{a} - \hat{a}^{\dagger})\right] \tag{30}$$

Similarly, adding Eqs. (24) and (25), we find

$$\hat{a} + \hat{a}^{\dagger} = 2\sqrt{\frac{m\omega}{2\hbar}}\,e^{\gamma t/2}\hat{x} \tag{31}$$

which gives the position operator as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,e^{-\gamma t/2}\,(\hat{a} + \hat{a}^{\dagger}) \tag{32}$$

Some mathematical analysis is required in order to calculate the position and momentum uncertainties, and hence their product, for the damped harmonic oscillator when the Generalized Uncertainty Principle (GUP) is applied. This allows us to examine how the Heisenberg uncertainty relation is modified.

The momentum operator obtained in the previous section can be written as

$$\hat{p} = \frac{1}{\sqrt{\beta}}\tan\left[-i\sqrt{\frac{\hbar m\omega\beta}{2}}\,e^{\gamma t/2}\,(\hat{a} - \hat{a}^{\dagger})\right] \tag{33}$$

Let us define

$$C = -i\sqrt{\frac{\hbar m\omega\beta}{2}}\,e^{\gamma t/2}, \qquad \hat{A} = \hat{a} - \hat{a}^{\dagger} \tag{34}$$

Using the Taylor series expansion of the tangent function, the momentum operator becomes

$$\hat{p} = \frac{1}{\sqrt{\beta}}\left[C\hat{A} + \frac{1}{3}(C\hat{A})^3 + \frac{2}{15}(C\hat{A})^5 + \cdots\right] \tag{35}$$

or equivalently,

$$\hat{p} = \frac{1}{\sqrt{\beta}}\,C\hat{A}\left[1 + \frac{1}{3}(C\hat{A})^2 + \frac{2}{15}(C\hat{A})^4 + \cdots\right] \tag{36}$$

Squaring the momentum operator, we obtain

$$\hat{p}^2 = \frac{(C\hat{A})^2}{\beta}\left[1 + \frac{2}{3}(C\hat{A})^2 + \frac{17}{45}(C\hat{A})^4 + \cdots\right] \tag{37}$$

The expectation values of the momentum operator and its square in the number state $|n\rangle$ are then given by

$$\langle\hat{p}\rangle_n = \frac{1}{\sqrt{\beta}}\left[C\langle n|\hat{A}|n\rangle + \frac{1}{3}C^3\langle n|\hat{A}^3|n\rangle + \frac{2}{15}C^5\langle n|\hat{A}^5|n\rangle\right] \tag{38}$$

$$\langle\hat{p}^2\rangle_n = \frac{C^2}{\beta}\left[\langle n|\hat{A}^2|n\rangle + \frac{2}{3}C^2\langle n|\hat{A}^4|n\rangle + \frac{17}{45}C^4\langle n|\hat{A}^6|n\rangle\right] \tag{39}$$

The momentum uncertainty is defined as

$$(\Delta p_n)^2 = \langle\hat{p}^2\rangle_n - \langle\hat{p}\rangle_n^2. \tag{40}$$

Substituting the expectation values of $\langle \hat{p} \rangle_n$ and $\langle \hat{p}^2 \rangle_n$, we obtain

$$(\Delta p_n)^2 = \frac{C^2}{\beta}(2n+1) + \frac{2}{3}C^4\, 3(2n^2+2n+1) - \frac{17}{45}C^6(10n^3+19n^2+26n+10). \tag{41}$$

The above expression represents the momentum uncertainty for the damped harmonic oscillator under the Generalized Uncertainty Principle. Similarly, the uncertainty in position is given by

$$(\Delta x_n)^2 = \langle \hat{x}^2 \rangle_n - \langle \hat{x} \rangle_n^2 \tag{42}$$

Using the position operator obtained earlier, we find

$$(\Delta x_n)^2 = \frac{\hbar}{2m\omega}\, e^{-\gamma t} \langle n|\hat{A}^2|n\rangle, \tag{43}$$

which gives

$$(\Delta x_n)^2 = \frac{\hbar}{2m\omega}\, e^{-\gamma t}(2n+1) \tag{44}$$

Therefore,

$$(\Delta x_n)^2 = \frac{\hbar}{m\omega}\, e^{-\gamma t}\left(n+\frac{1}{2}\right) \tag{45}$$

We now evaluate the uncertainty product of position and momentum:

$$\Delta p_n\, \Delta x_n = \hbar\left(n+\frac{1}{2}\right)\left[1 + \frac{e^{\gamma t}\beta\hbar m\omega}{2}\,\frac{2n^2+2n+1}{2n+1} + \frac{17\beta^2}{180}\, e^{2\gamma t}(\hbar m\omega)^2 \times \frac{10n^3+19n^2+26n+10}{2n+1}\right]^{1/2}. \tag{46}$$

Clearly, by setting $\beta = 0$, we recover the usual Heisenberg uncertainty relation corresponding to the simple harmonic oscillator. In the above expression, the parameter β encodes the effects of the Generalized Uncertainty Principle, while the exponential time-dependent factors reflect the presence of damping in the harmonic motion.

6. Graphical Representation of the Uncertainty Products

The generalized uncertainty principle (GUP) may be written as [5]

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}\left[1 + \beta(\Delta p)^2 + \gamma\right] \tag{47}$$

For simplicity, we set $\gamma = 0$, retaining only the β-dependent correction. The inequality then reduces to

$$\Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p} + \beta \Delta p\right) \tag{48}$$

This relation represents a hyperbola-like curve in the $(\Delta p, \Delta x)$ plane, exhibiting a minimum uncertainty in position. The minimum occurs at

$$\Delta x_{\min} = \hbar\sqrt{\beta} \tag{49}$$

with the corresponding momentum uncertainty

$$\Delta p = \frac{1}{\sqrt{\beta}}. \tag{50}$$

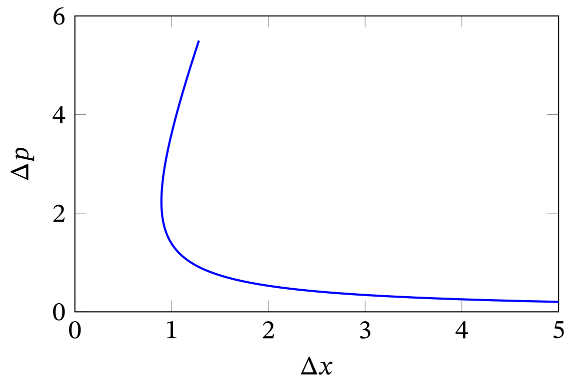

Figure 1. Modified uncertainty relation: $\Delta x\, \Delta p \geq \frac{\hbar}{2}(1+\beta(\Delta p)^2)$, showing minimal $\Delta x_0 > 0$.

For the damped harmonic oscillator, the minimum uncertainty becomes

$$(\Delta x\, \Delta p)_{\min} = \frac{\hbar}{2}(2n+1) \tag{51}$$

When damping is included, the correction acquires a time-dependent factor:

$$\Delta x\, \Delta p \gtrsim \hbar N\left[1 + \beta e^{\gamma t} F_1(n) + \ldots\right] \tag{52}$$

where, $N = \left(n+\frac{1}{2}\right)$. The above expression we've found from the previous section. But for graphical representation, we will use the following expression in which Δx is a function of Δp and time, t

$$\Delta x(\Delta p, t) \approx \hbar N\left(\frac{1}{\Delta p} + \beta e^{\gamma t}\Delta p\right) \tag{53}$$

i. Time evolution of the curve

- At $t = 0$, it reduces to the usual GUP curve.
- For $t > 0$, the β-term is multiplied by $e^{\gamma t}$, pushing the curve upward.

The minimal position uncertainty becomes:

$$(\Delta x)_{\min}(t) \approx 2\hbar N\sqrt{\beta}\, e^{\frac{1}{2}\gamma t} \tag{54}$$

and the corresponding optimal momentum uncertainty shifts to:

$$(\Delta p)_{\text{opt}}(t) = \frac{1}{\sqrt{\beta}}\, e^{-\frac{1}{2}\gamma t}. \tag{55}$$

ii. Graphical picture

If we plot Δp vs Δx:

- At $t = 0$, we have the standard curve with minimum at $(\Delta p = 1/\sqrt{\beta},\ \Delta x = 2\hbar N\sqrt{\beta})$.

- As t grows, the curve stretches upward (larger Δx) and shifts leftward (smaller optimal Δp).

The allowed region still lies above the curve, but the forbidden region widens near small Δx.
Visually, the U-shaped curve "slides up and left" with time due to damping.

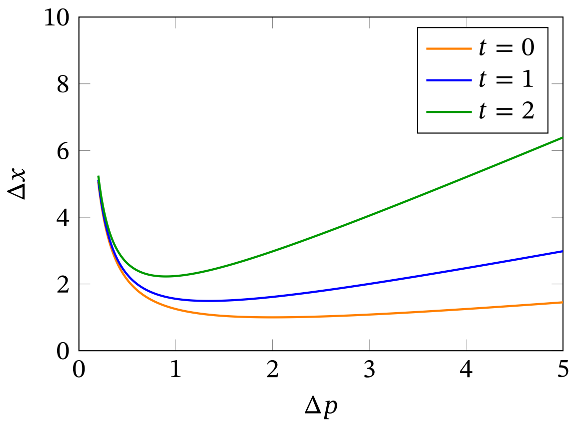

Figure 2. Time evolution of the Generalized Uncertainty Principle curve for different values with time damping.

7. Conclusion

The final expression in section 5 reveals the uncertainty relation for a damped harmonic oscillator under the minimal length consideration, which is GUP. The uncertainty relation is no longer depends linearly on the number state for quantum harmonic oscillator. Graphical representation shows the variation in minimal length as time evolve. This gives a total insight of both the time evolution of the operators and the variation of the standard deviation of the position and momentum operators with number states in a new way.

References

[1] J. Guerrero and F. F. López-Ruiz, "On the Lewis–Riesenfeld (Dodonov–Man'ko) invariant method," *Phys. Scr.*, vol. 90, no. 7, p. 074046, 2015. https://doi.org/10.1088/0031-8949/90/7/074046. [Online]. Available: https://arxiv.org/abs/1503.01371.

[2] P. N. Swamy, "Generalized Heisenberg relation and quantum harmonic oscillators," Dept. of Physics, Southern Illinois Univ., Edwardsville, IL. [Online]. Available: https://arxiv.org/abs/quant-ph/0610079.

[3] A. Camacho, "Some consequences of a generalization to Heisenberg algebra in quantum electrodynamics," Physics Dept., Univ. Autónoma Metropolitana–Iztapalapa, Mexico. [Online]. Available: https://arxiv.org/abs/gr-qc/0305052.

[4] T. Lancaster and S. J. Blundell, *Quantum Field Theory for the Gifted Amateur*. Oxford, U.K.: Oxford Univ. Press, 2014.

[5] A. Kempf, G. Mangano, and R. B. Mann, "Hilbert space representation of the minimal length uncertainty relation," *Phys. Rev. D*, vol. 52, pp. 1108–1118, 1995. https://doi.org/10.1103/PhysRevD.52.1108. [Online]. Available: https://arxiv.org/abs/hep-th/9412167.